223

Army worms, etc. (Betts)

Cornell Co-operative Society

Ask For No. 880-CBXM.

Spiral Binding—U S. Patents 1516932-1942026. Other Patents Pending.

Physics 223 — Electric Waves. — Prof. Bethe.

Sept 26. References:

Vilbig — Hochfrequenztechnik. Good up to 1937

Since 1937: much development; esp. resonant cavities

Hansen — Jour. App. Phys.

: directed beams;

Course:

Alternating currents; quasi-stationary treatment;

Transmission lines;

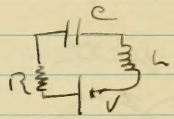
Waves along conductors —

Emission from antennae.

Quasi-Stationary treatment.

Assume I, V function of time, + connected by circuit equation.

e.g. simple RLC circuit



$$V = RI + \frac{q}{C} + L\frac{dI}{dt}. \quad \text{(electromagnetic units)}$$

$$I = \frac{dq}{dt}.$$

Important case: periodic time dependence of V.

[if V is not sinusoidal, can be Fourier-analyzed anyhow, or each component treated separately].

Hence assume. $V = V_0 e^{i\omega t}$.

[Can introduce any phase desired by allowing V_0 to be complex].

Then $I = I_0 e^{i\omega t}$.

$$q = \frac{I_0}{i\omega} e^{i\omega t}.$$

Whence $V_0 = I_0\left(R - \frac{i}{C\omega} + Li\omega\right)$

$$\therefore I_0 = \frac{V_0}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$

$$= \frac{V_0\left(R - i\left(\omega L - \frac{1}{\omega C}\right)\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

Or, let $R + i\left(\omega L - \frac{1}{\omega C}\right) = \mathcal{R}$. ("impedance")

(X = "reactance") $\mathcal{R} = R + iX$

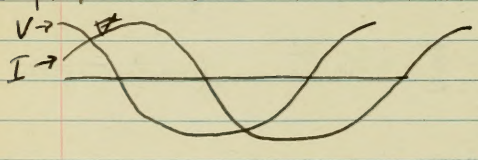
$$= \sqrt{R^2 + X^2}\, e^{i\tan^{-1}\frac{X}{R}} = R_0 e^{i\varphi}$$

Then if V_0 is real, $V = V_0 \cos\omega t$

$$I = \left(R.P.\left(\frac{V_0}{R_0} e^{i(\omega t - \varphi)}\right)\right)$$

$$= \frac{V_0}{R_0}\cos(\omega t - \varphi).$$

φ positive:



Energy dissipated $= \overline{VI} = \frac{V_0^2}{R_0}\overline{\cos\omega t\cos(\omega t - \varphi)}$

$$= \frac{V_0^2}{R_0}\overline{\cos\omega t\cos\omega t\cos\varphi + \cos\omega t\sin\omega t\sin\varphi}$$

$$= \frac{1}{2}\frac{V_0^2}{R_0}\cos\varphi.$$

But $\frac{V_0}{R_0} = |I_0|$

$\therefore \overline{VI} = \frac{1}{2}|I_0|^2 R_0 \cos\phi$

But $R_0 \cos\phi$ = real part of R = R.

$\therefore \overline{VI} = \frac{1}{2}|I_0|^2 R$.

VI was not always positive; sometimes negative. This because besides Joule heat, there is electric energy in C, & magnetic energy in inductance; $W_{el} = \frac{q^2}{2C}$, $W_{mag} = \frac{1}{2}LI^2$; these increase & decrease periodically (building up & decay of el. & mag. fields) But Joule heat never comes back.

$$\frac{dW_{el}}{dt} = \frac{q}{C}I \quad ; \quad \frac{dW_{mag}}{dt} = LI\frac{dI}{dt}.$$

$$\frac{d}{dt}(W_{el} + W_{mag}) = I\left(\frac{q}{C} + L\frac{di}{dt}\right)$$

add Joule heat: $I \cdot IR$.

$$\text{Total} = I\left(\frac{q}{C} + L\frac{di}{dt} + IR\right) = IV.$$

Resonance : $\bar{X} = 0$.

or : $\omega L = \frac{1}{\omega C}$.

$\omega = (LC)^{-1/2}$.

Dimensions : $L \sim l$, $C \sim l^{-1} t^2$

$\therefore \omega \sim (l l^{-1} t^2)^{-1/2} = t^{-1}$.

Wave length emitted to free space : $\lambda = \frac{2\pi c}{\omega}$

$= 2\pi c \sqrt{LC}$.

But $C_{em} = \frac{C_{e.s.}}{c^2}$

hence $\lambda = 2\pi \sqrt{L\, C_{e.s.}}$

$L \sim l$, $C_{e.s.} \sim l$, $\therefore \lambda$ obviously $\sim l$.

At Resonance R_0 is minimum , $|I_0|$ maximum .

and $\varphi = 0$, $\therefore V$ + I are in phase .

If $R \ll \sqrt{\frac{L}{C}}$, very pronounced minim in R_0 ,

" " maxim in $|I_0|$.

i.e. strong resonance .

Sept 29 | Want to investigate sharpness of minimum in R_0.

$$R_0^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 .$$

$$\omega_{res} L \frac{\omega}{\omega_{res}} - \frac{1}{C}\frac{\omega_{res}}{\omega\,\omega_{res}} = \sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right)$$

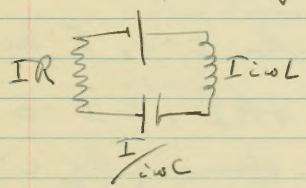
$$= \sqrt{\frac{L}{C}}\left(\frac{\omega^2 - \omega_{res}^2}{\omega\,\omega_{res}}\right).$$

If ω is of same order of mag. as ω_{res}, but different, then reactance $\sim \sqrt{\frac{L}{C}}$.

$\therefore$ Resonance is sharp if $R \ll \sqrt{\frac{L}{C}}$.

$\therefore$ define $Q = \frac{\sqrt{L/C}}{R}$; large Q means sharp resonance

Consider Physical meaning of $\sqrt{L/C}$:



EMF across L at resonant frequency
$= I\omega_{res} L = I\sqrt{\frac{L}{C}}$.

EMF across C at resonant frequency
$= I/\omega_{res} C = I\sqrt{\frac{L}{C}}$.

$\therefore \sqrt{\frac{L}{C}}$ is the apparent resistance of either L or C at the resonant frequency. If large Q, then large E.M.F. across both L & C at resonance, but they cancel each other —

Numerical example: Suppose we want a frequency $10^5 = 100$ kc.

then $\omega = 2\pi f = 6.3 \cdot 10^5 \text{ sec}^{-1}$.

henry $= 10^9$ cm.
1 farad $= 9 \cdot 10^{11}$ cm.

$$LC = \frac{1}{\omega^2} = \frac{1}{4 \cdot 10^{10}} = .25 \cdot 10^{-10}$$

Suppose	$C =$	1,	.10,	.01 μf and.
then	$L =$	2.5	25	250 μhenry.
and $Q = \sqrt{L/C}$	$=$	1.6	16	160 ohms.

for L, need coil, which has R; for a coil of n turns, $L \sim n^2$, $R \sim n$; ∴ high L, low C is better, in general for high Q.

Free oscillations

Suppose V is turned off when $t=0$, I known.

then subsequently: $q/C + RI + L\dot{I} = 0$.

assume $q = q_0 e^{\alpha t}$

then $\frac{1}{C} + R\alpha + L\alpha^2 = 0$

$$\alpha = -\frac{R}{2L} \pm \frac{\sqrt{R^2 - 4L/C}}{2L}$$

$$= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$$

Two solutions; 3 cases, $\sqrt{\ }$ real, 0 or imag.

a) $\frac{1}{CL} > \frac{R^2}{4L^2}$, or $R < 2\sqrt{\frac{L}{C}}$, or $Q > \frac{1}{2}$.

then two solutions: $\alpha = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

damped oscillations. $= -\frac{R}{2L} \pm i\omega_0$.

$$q \sim e^{\pm i\omega_0 t - \frac{Rt}{2L}}.$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \omega_{res}\sqrt{1 - \frac{1}{4Q^2}}.$$

$$\frac{R}{2L} = \omega_{res}\frac{R}{2\sqrt{L/C}} = \frac{\omega_{res}}{2Q}.$$

$$\therefore q \sim e^{\pm i\omega_0 t - \frac{\omega_{res} t}{2Q}}.$$

Damping for one period: $\frac{2\pi}{\omega_0} \cdot \frac{\omega_{res}}{2Q}$

$$= \frac{\pi}{Q}\frac{1}{\sqrt{1 - \frac{1}{4Q^2}}} = \frac{\pi}{\sqrt{Q^2 - \frac{1}{4}}}$$

For large Q, damping $= \pi/Q$; for small damping, need high Q.

b) $\frac{R^2}{4L^2} > \frac{1}{LC}$. $R > 2\sqrt{\frac{L}{C}} = Q < \frac{1}{2}$.

Two roots: $\alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$.

$$= -\frac{R}{2L}\left\{1 \mp \sqrt{1 - 4Q^2}\right\}$$

if $Q \ll 1$, $|\alpha_1| = -\frac{R}{L}$, $\alpha_1\alpha_2 = \frac{1}{CL}$, hence $|\alpha_2| = \frac{1}{CR}$

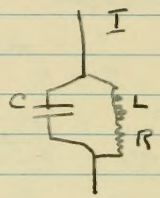
$q = a_1 e^{-|\alpha_1|t} + a_2 e^{-|\alpha_2|t}$ at large t, $q \sim a e^{-|\alpha_2|t}$.

e.g. $C \sim 10^{-8}$ farad, $R = 1\,\Omega$, $|\alpha_1| = 10^8\ \text{sec}^{-1}$

$C \sim 10^{-6}$ farad, $R = 1$ meg, $|\alpha_1| = 1\ \text{sec}^{-1}$.

c) $\frac{1}{LC} = \frac{R^2}{4L^2}$, one sol'n, $q = e^{-\frac{R}{2L}t}(q_1 + q_2 t)$.

Oct. 1 Consider:



$$\frac{1}{\mathcal{R}} = \frac{1}{1/i\omega C} + \frac{1}{R + i\omega L}$$

$$G = i\omega C + \frac{1}{R + i\omega L}$$

$$= \frac{i\omega C R - \omega^2 L C + 1}{\cancel{R^2 + \omega^2 L^2}\ R + i\omega L}$$

Define $G = \frac{1}{\mathcal{R}}$; conductance.

$$\therefore \mathcal{R} = \frac{R + i\omega L}{1 - \omega^2 LC + iRC\omega}$$

$$\mathcal{R} = \frac{(R + i\omega L)(1 - \omega^2 LC - iRC\omega)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

$$= \frac{R(1 - \omega^2 LC) + \omega^2 RLC + i\left(\omega L(1 - \omega^2 LC) - \omega R^2 C\right)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

$$= \frac{R + i\omega\left(L - \omega^2 L^2 C - R^2 C\right)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

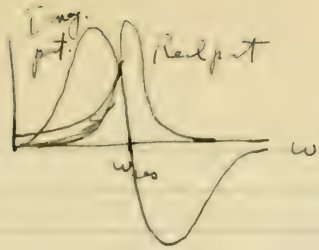
$$\omega_{res} = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\sqrt{L/C}}{R}$$

$$= \frac{R + i\omega L\left(1 - \frac{\omega^2}{\omega_{res}^2} - \frac{1}{Q^2}\right)}{\left(1 - \frac{\omega^2}{\omega_{res}^2}\right)^2 + \left(\frac{\omega}{Q\omega_{res}}\right)^2}$$

Real part of Impedance has sharp minimum at resonant frequency.

Imaginary part goes thru zero at resonant frequency —



analogous to dispersion theory : R.P. to absorption,
I.P. to dispersion.

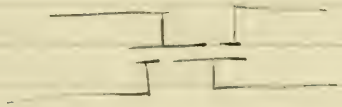
This must be taken if a particular frequency is to be not out of resonance circuit.

Coupled Circuits.

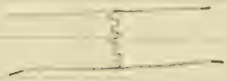
can have: Inductive coupling
Capacitive "
Resistive "

Inductive :

Capacitive :



Resistive



Inductive most common.

Mutual inductance M.

Equations of Inductively-coupled circuits:

$$L_1 \dot{I}_1 + R_1 I_1 + \int_1 C_1^{-1} + M \dot{I}_2 = 0$$
$$L_2 \dot{I}_2 + R_2 I_2 + \int_2 C_2^{-1} + M \dot{I}_1 = 0$$

(free oscillation)

Solutions of form: $I_1 = a_1 e^{\alpha t}$
$I_2 = a_2 e^{\alpha t}$

$$a_1\left(L_1\alpha + R_1 + \frac{1}{C_1\alpha}\right) + a_2\alpha M = 0$$
$$a_1\alpha M + a_2\left(L_2\alpha + R_2 + \frac{1}{C_2\alpha}\right) = 0$$

Hom. homog. eq.; have sol'n only if $D\phi = 0$;

$$\therefore \left(L_1\alpha + R_1 + \frac{1}{C_1\alpha}\right)\left(L_2\alpha + R_2 + \frac{1}{C_2\alpha}\right) - \alpha^2 M^2 = 0.$$

Fourth order; in general four roots for α.

Special case:

1) M small (close coupling).
In particular, $M^2 \ll L_1 L_2$.

Then $L_{1(2)}\alpha + R_{1(2)} + \frac{1}{C_{1(2)}\alpha} \cong 0$

$$\therefore \alpha_{1(2)}' \cong -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Also α_1'', α_1''; α_2', α_2''.

Actual solutions are $\alpha = \alpha_1' + \beta_1'$ $\beta_1' \ll \alpha_1'$.

Then $\frac{1}{C_1\alpha} + R_1 + L_1\alpha = \beta_1'\left(\frac{1}{\alpha}(\ \ )\right) = \beta_1'\left(L_1 - \frac{1}{C_1\alpha^2}\right)$

$$\therefore \quad \beta_1'\left(L_1 - \frac{1}{C_1\alpha_1'^2}\right) = \frac{M^2\alpha_1'^2}{\left(\frac{1}{C_2\alpha_1'} + R_2 + L_2\alpha_1'\right)}$$

Hence we can solve for β_1'.

If R_1 is very small, $\alpha_1' = \pm i\,\omega_{res} = \pm \frac{i}{\sqrt{L_1 C_1}}$,

and $L_1 - \frac{1}{C_1\alpha_1'^2} = L_1 - \frac{1}{C_1\left(-\frac{1}{L_1C_1}\right)} = 2L_1$

and $\beta_1' = \frac{M^2\alpha_1'^2}{2L_1\left(L_2\alpha_1' + R_2 + \frac{1}{C_2\alpha_1'}\right)}$.

Same matter for α_1'' and β_1'' ($\alpha_1'', \alpha_2', \alpha_2''$)

Oct. 2

Problem 1. Series C-L-R circuit.

L: 100 turns
1 cm. diameter
4 cm. length.
wire diameter = .1 cm, Cu.

Calculate L & R.
R of coil is only R.

Want res. frequency = 1 Mc = 10^6 cycle/sec. ($\omega = 2\pi f$)

What capacitance is necessary? What is Q?

Assume solution α from above; M zero, $\alpha_1', \alpha_1'', \alpha_2', \alpha_2''$;

$$L_1\alpha^2 + R_1\alpha + \frac{1}{C_1} = (\alpha - \alpha_1')(\alpha - \alpha_1'')L$$

Same for α_2;

Hence Fundamental equation becomes:

$$(\alpha-\alpha_1')(\alpha-\alpha_1'')(\alpha-\alpha_2')(\alpha-\alpha_2'') = \frac{M^2}{L_1 L_2}\alpha^4 .$$

$\frac{M^2}{L_1 L_2}$ is called (coupling coefficient)^2; "ideally" $= 1$ when coils with long no of turns inside each other, so that field of one coil is concentrated inside it;

$$k^2 = \frac{M^2}{L_1 L_2} . \qquad k = \text{coupling coeff.}$$

Weak coupling; k^2 small; hence at least one factor on left above must be small. or:

Each "eigenvalue" must be close to an "unperturbed eigenvalue" α_1'' α_2', α_2'' α_2'.

Look for one close to α_1': $\alpha \cong \alpha_1'$

Hence $(\alpha-\alpha_1) = \dfrac{k^2\alpha_1'^4}{(\alpha_1'-\alpha_1'')(\alpha_1'-\alpha_2')(\alpha_1'-\alpha_2'')}$

OK. if none of factors in denom. vanishes.

$\alpha_1'-\alpha_1''$ will in general not vanish, esp. in true oscillating circuit. (circuit 1) i.e. if circuit 1 has large Q; then $\alpha_1'-\alpha_1'' = 2i\omega$; for large Q, $\alpha_1' = i\omega_0 - \beta_1$, $\beta_1 \simeq \frac{\omega_0}{2Q}$, if large Q, $\beta_1 \ll \omega_0$; only if $Q = \frac{1}{2}$ do we get double root.

$\alpha_1'-\alpha_2'$ might vanish; esp. if identical circuits, or if same frequency + damping; i.e. if they are "tuned" to each other. Also possible

to have $\alpha_1'' = \alpha_2''$. For Tuned circuit, formula breaks down.

For Tuned circuits, : $\alpha_1' \cong \alpha_2'$, $\alpha_1'' \cong \alpha_2''$;

Hence original eq. becomes:

$$(\alpha - \alpha_1')^2 (\alpha - \alpha_1'')^2 = k^2 \alpha^4 ;$$

$$\therefore \quad \alpha - \alpha_1 = \pm \frac{k \alpha^2}{\alpha - \alpha_1''} .$$

For Untuned circuit : $\Delta\alpha\,(\alpha - \alpha_1') \sim k^2$

" Tuned " : $\Delta\alpha \sim k$; also ambiguity of sign

i.e. new freq. may be either larger or smaller.

i.e. eigenvalue is split by the coupling.

Untuned circuits are non-degenerate (i.e. only current 1 responds to α_1) and perturbation is prop. to square of coupling; Tuned circuits are degenerate, & perturbation is ~~prop. to~~ linear in coupling.

Relative amplitudes : $I_1 = a_1 e^{\alpha t}$ $I_2 = a_2 e^{\alpha t}$;

then :
$$\begin{cases} (L_1 \alpha^2 + R_1 \alpha + \frac{1}{C_1}) a_1 + M \alpha^2 a_2 = 0 \\ M \alpha^2 a_1 + (L_2 \alpha^2 + R_2 + \frac{1}{C_2}) a_2 = 0 \end{cases}$$

$$(\alpha - \alpha_1')(\alpha - \alpha_1'') + \frac{M}{L_1} \alpha^2 a_2 = 0$$

$$\downarrow$$

$$k \sqrt{\frac{L_2}{L_1}}\, \alpha^2 a_2$$

or
$$\begin{cases} (\alpha - \alpha_1)'(\alpha - \alpha_1'') \sqrt{L_1}\, a_1 + k \alpha^2 \sqrt{L_2}\, a_2 = 0 \\ (\alpha - \alpha_2)'(\alpha - \alpha_2'') \sqrt{L_2}\, a_2 + k \alpha^2 \sqrt{L_1}\, a_1 = 0 \end{cases}$$

~~If k is small,~~

Non-degenerate case (mutual circuits):

Take the α near α_1'; $(\alpha - \alpha_1') \sim k^2$

$\therefore$ Equation becomes:

$$a_1 \sqrt{L_1}\, k^2 + a_2 \sqrt{L_2}\, k \cong 0$$

$$\text{or } \frac{a_2 \sqrt{L_2}}{a_1 \sqrt{L_1}} \sim k.$$

$$\left(\text{Exactly: } \frac{k\, \alpha_1'^2}{(\alpha_1' - \alpha_2')(\alpha_1' - \alpha_2'')}\right)$$

$\therefore$ $a_2 \ll a_1$. i.e. current I expands, since frequency is near its current freq; second current was current prop. to k - coupling coeff. Also, from exact expression, the more out of tune the circuits are, $(\alpha_1' - \alpha_2')$ large, very small response by second circuit.

If tune $\alpha \sim \alpha_2'$, opposite happens.

Degenerate case (tuned circuits;)

Now $(\alpha - \alpha_1') \sim k$;

i.e. Both sides of expression are of order k, hence same order of magnitude.

$$(\alpha_1 - \alpha_1') = \pm \frac{k\, \alpha_1'^2}{\alpha_1' - \alpha_1''}$$

$$\therefore \pm k\, \alpha_1'^2 \sqrt{L_1}\, a_1 + k\, \alpha_1'^2 \sqrt{L_2}\, a_2 = 0$$

$$\therefore \frac{a_1}{a_2} = \pm \sqrt{\frac{L_1}{L_2}}.$$

Amplitudes are equal if amplitude is defined as $(a\sqrt{L})$; either the same or opposite sign.

Two "modes" ; one frequency corr. to each one.

$\alpha_1' = i\omega, \quad \alpha_2'' = -i\omega$;

$$\text{then} \quad \alpha - i\omega = \pm \frac{k_2 (i\omega)^2}{2 i \omega}$$

$$= \pm \, i \tfrac{1}{2} k \omega \, .$$

$$\therefore \alpha = i\omega\left(1 \pm \tfrac{1}{2} k\right) .$$

higher freq. corr. to current in same direction;
lower " " " " " opposite " .

Oct. 6 '41

Case of Strong coupling : $k \approx 1$

[k can never be greater than 1 ; to see this, consider Magnetic Energy $\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$; this is gotten from $U = \frac{1}{8\pi} \int H^2 d\tau$; since U is positive, Mag. Energy must be +, even tho I_1 + I_2 be opposite ; Now write mag energy as : $\frac{1}{2}(\sqrt{L_1} I_1)^2 + \frac{1}{2}(\sqrt{L_2} I_2)^2 \mp (\sqrt{L_1} I_1)(\sqrt{L_2} I_2)$

if $\sqrt{L_1} I_1 = \sqrt{L_2} I_2$, energy $= L_1 I_1^2 (1 \mp k)$; hence $k \leq 1$.

Or: Mag en. $= \frac{1}{2}\left[(\sqrt{L_1} I_1 - \sqrt{L_2} I_2)^2 + 2(1-k)\sqrt{L_1 L_2}\, I_1 I_2\right]$ & same result.] .

Equation for characteristic frequencies was :

$$\left(L_1\alpha^2 + R_1\alpha + \frac{1}{C_1}\right)\left(L_2\alpha^2 + R_2\alpha + \frac{1}{C_2}\right) = M^2\alpha^4 .$$

Assume $R_1 \ll \sqrt{\frac{L_1}{C_1}}$, $R_2 \ll \sqrt{\frac{L_2}{C_2}}$; i.e. $Q_1 \gg 1$, $Q_2 \gg 1$

Hence neglect R_1, R_2 : $\left(L_1\alpha^2 + \frac{1}{C_1}\right)\left(L_2\alpha^2 + \frac{1}{C_2}\right) = M^2\alpha^4$;

or : $(L_1L_2 - M^2)\alpha^4 + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1}\right)\alpha^2 + \frac{1}{C_1C_2} = 0$

$k^2 = \frac{M^2}{L_1L_2}$ $\qquad = L_1L_2(1-k^2)\alpha^4 + \frac{L_1C_1 + L_2C_2}{C_1C_2}\alpha^2 + \frac{1}{C_1C_2} = 0$

or: $(1-k^2)\alpha^4 + \left(\frac{1}{L_2C_2} + \frac{1}{L_1C_1}\right)\alpha^2 + \frac{1}{L_1C_1L_2C_2} = 0$

$\omega_1^2 = \frac{1}{L_1C_1}$, $\omega_2^2 = \frac{1}{L_2C_2}$: $(1-k^2)\alpha^4 + (\omega_1^2 + \omega_2^2)\alpha^2 + \omega_1\omega_2 = 0$

Whence $\alpha^2 = \frac{-(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2+\omega_2^2)^2 - 4\omega_1\omega_2(1-k^2)}}{2(1-k^2)}$

since second term is less than first, both solutions are negative :

$\therefore$ write $\omega^2 = -\alpha^2 = \frac{\omega_1^2+\omega_2^2 \pm \sqrt{(\omega_1^2-\omega_2^2)^2 + 4k^2\omega_1^2\omega_2^2}}{2(1-k^2)}$

[For small k, easily verified that same results as before obtain —].

For Tuned circuits : $\omega_1 = \omega_2$:

$$\omega^2 = \frac{2\omega_1^2 \pm 2k\omega_1^2}{2(1-k^2)} = \frac{\omega_1^2}{1 \pm k}$$

or $\omega = \frac{\omega_1}{\sqrt{1 \pm k}}$.

as $k \to 1$, the two frequencies approach $\frac{\omega_1}{\sqrt{2}}$ & ∞, resp.

Amplitudes :

$$a_1\left(L_1\alpha^2 + \frac{1}{C_1}\right) + M a_2 \alpha^2 = 0$$

$$a_1 \sqrt{L_1}\left(\alpha^2 + \frac{1}{L_1 C_1}\right) + k\sqrt{L_1 L_2}\,\alpha^2 a_2 = 0$$

$$a_1 \sqrt{L_1}\left(-\omega^2 + \omega_1^2\right) - k\sqrt{L_2}\, a_2\, \omega^2 = 0$$

$$\therefore \quad \frac{a_2\sqrt{L_2}}{a_1\sqrt{L_1}} = \frac{\omega_1^2 - \omega^2}{k\omega^2}$$

Tuned circuits : $\omega^2 = \dfrac{\omega_1^2}{1 \pm k}$:

$$\therefore \quad \frac{a_2\sqrt{L_2}}{a_1\sqrt{L_1}} = \frac{\omega^2(1 \pm k) - \omega^2}{k\omega^2}$$

$$= \pm 1 .$$

Hence equal amplitude ; in phase for low frequency, out of phase for high freq.

Criterion for tuning : from equation for ω^2 :

Circuits tuned if : $(\omega_1^2 - \omega_2^2) < 2k\,\omega_1\omega_2$;

Or, if k is not too large, $(\omega_1^2 - \omega_2^2) < k(\omega_1 + \omega_2)^2 \cdot \frac{1}{2}$

or $\dfrac{\omega_1 - \omega_2}{\frac{1}{2}(\omega_1 + \omega_2)} < k$.

i.e. difference in freq. tolerable by average freq. $< k$.

Let ω', ω'' be frequencies of coupled circuits ;

then $\omega'^2 - \omega''^2 = \dfrac{\sqrt{(\omega_1^2 - \omega_2^2)^2 + 4k^2\omega_1^2\omega_2^2}}{(1 - k^2)} > |\omega_1^2 - \omega_2^2|$

i.e. difference in frequencies of system coupled,

is greater the difference in natural frequencies; (just as in Q-M., a perturbation always pushes almost degenerate levels farther apart).

Problem 2. Find damping in case of strong coupling.
By: Treat resistance as a perturbation; say actual solution is close to sol'n's without resistance. (Use Q's).

Forced oscillations of coupled circuits.
Assume Circuit 1 has E.M.F. (sinusoidal).
then:

$$\begin{cases} L_1 \dot{I}_1 + R_1 I_1 + \frac{q_1}{C_1} + M \dot{I}_2 = E = E_0 e^{i\omega t} \\ L_2 \dot{I}_2 + R_2 I_2 + \frac{q_2}{C_2} + M \dot{I}_1 = 0 \end{cases}$$

Put $I_1 = a_1 e^{i\omega t}$, $I_2 = a_2 e^{i\omega t}$

then:

$$\begin{cases} a_1\left(i\omega L_1 + R_1 - \frac{i}{\omega C_1}\right) + a_2 M i\omega = E_0 \\ a_2\left(i\omega L_2 + R_2 - \frac{i}{\omega C_2}\right) + a_1 M i\omega = 0 \end{cases}$$

or:

$$\begin{cases} a_1\left(-\omega^2 L_1 + i\omega R_1 + \frac{1}{C_1}\right) - a_2 \omega^2 M = i\omega E_0 \\ -a_1 \omega^2 M + a_2\left(-\omega^2 L_2 + i\omega R_2 + \frac{1}{C_2}\right) = 0 \end{cases}$$

solve for a_1:

$$a_1\left\{\left(-L_1\omega^2 + i\omega R_1 - \frac{1}{C_1}\right)\left(-L_2\omega^2 + iR_2\omega + \frac{1}{C_2}\right) - M^2\omega^4\right\} = iE_0\omega\left(-L_2\omega^2 + iR_2\omega + \frac{1}{C_2}\right)$$

$\{\ \}$: characteristic quantity for free osc.
$(\ )$: " " " " " " 2nd circuit.

Let $\omega_1\ \omega_2\ \omega_3\ \omega_4$ be char. freq. of coupled circuit.
" ω_5; ω_6 " " " " " circuit 2:

then : $(L_1 L_2 - M^2)(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)\, a_1$

$$= -i E_0 \omega L_2 (\omega-\omega_5)(\omega-\omega_6)$$

∴ Resonance (a_1 max.) if $\omega \approx$ one of chan freq. of coupled circuit; but a_1 vanishes if $\omega \approx$ one of freq. of 2nd circuit!

For small R_1, R_2, $\omega_3 \sim -\omega_1$, $\omega_4 \sim -\omega_2$;

$\omega_6 \sim \omega_5$

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$\omega_5 = \omega_5' + i\beta_5$ $\quad$ $\omega_1 = \omega_1' + i\beta_1$ $\quad$ $\omega_2 = \omega_2' + i\beta_2$

$\omega_6 = -\omega_5' + i\beta_5$ $\quad$ $\omega_3 = -\omega_1' + i\beta$ $\quad$ $\omega_4 = -\omega_2' + i\beta_2$

then $$a_1 = \frac{-i E_0 \omega L_2}{L_1 L_2 - M^2} \cdot \frac{(\omega-\omega_5'-i\beta_5)(\omega+\omega_5'-i\beta_5)}{(\omega-\omega_1'-i\beta_1)(\omega+\omega_1'-i\beta_1)(\omega-\omega_2'-i\beta_2)(\omega+\omega_2'-i\beta_2)}$$

Resonances : ω_1', ω_2' large current

ω_5' small current (large E.M.F.)

(This combines features of series + parallel resonance)

series $\omega_1' \; \omega_2'$ $\quad$ parallel ω_5'

Here we have:

① can have series resonance at its own at ω_1' & ω_2';

② has EMF impressed on it from outside (like a ∥ series; so ∥ resonance at ω_5'.

$$\frac{a_2}{a_1} = \frac{i^2 M \omega^2}{(-\omega^2 L_2 + i R_2 \omega + \frac{1}{C_2})} - \frac{+E M \omega^2}{(\omega-\omega_5)(\omega-\omega_6)}$$

$$\therefore a_2 \sim \frac{M\omega^2}{(\omega-\omega_1)(\omega-\omega_2)(\;-\;)\cdot(\;)}$$

i.e. series resonance for ω_1', ω_2'.

More complicated circuits get too complicated; sometimes do them graphically – cf. Villey –.

<u>Transformer</u>. example of coupled circuit.

Usually use smaller frequencies than resonance; don't use resonance properties.

For $\omega \gg \omega_{\text{s}}$, $\quad \frac{a_2}{a_1} \simeq -\frac{M}{L}$.

" $\omega \ll \omega_{\text{s}}$ $\quad \frac{a_2}{a_1} \simeq M C_2 \omega^2$.

If $\omega \ll \omega_1', \omega_2', \omega_3'$, $\quad a_1 \cdot \frac{1}{C_1 C_2} = \frac{i\omega E_0}{C_2}$

$$\therefore a_1 \simeq i\omega E_0 C_1,$$

$$a_2 \simeq i\omega^3 E_0 C_1 C_2 M.$$

Potential diff in second circuit: $\frac{a_2}{i C_2 \omega} = \omega^2 E_0 C_1 M$.

For $\frac{1}{C_1}, \frac{1}{C_2}$ small, resonant frequencies very small;

then $\omega \gg \omega_1', \omega_2', \omega_3'$ (the normal case)

and $a_1 = -\frac{i E_0}{\omega} \frac{L_2}{L_1 L_2 - M^2} \simeq \frac{E_0}{i L_1 \omega} \frac{1}{(1-k^2)}$

$$a_2 = -\frac{i E_0}{\omega} \frac{M}{L_1 L_2 - M^2} \simeq -\frac{M}{L_2} \frac{E_0}{\omega L_1} \frac{1}{1-k^2}$$

For strong coupling, $M^2 \sim L_1 L_2$

and $-\frac{a_2}{a_1} = \frac{M}{L_2} \sim \sqrt{\frac{L_1}{L_2}}$.

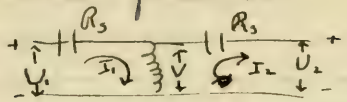
~~Pot. diff: $i\omega L_2 a_2 = E_0 L_2 \frac{M}{L_1 L_2 - M^2}$~~

Since $L \sim n^2$, $\quad \frac{a_2}{a_1} \sim \sqrt{\frac{L_1}{L_2}} \sim \frac{n_1}{n_2}$.

Filters. circuits which will not (or will) transmit certain frequencies.

Can make filters so:

E.M.F. u_1



R_c = impedance of coil;

$$V = (I_1 - I_2) R_c$$
$$U_1 = V + I_1 R_s$$
$$U_2 = V - I_2 R_s$$

Want I_2, U_2 in terms of I_1, U_1.

$$-I_2 = \frac{V}{R_c} - I_1$$
$$= \frac{U_1 - I_1 R_s}{R_c} - I_1$$
$$I_2 = -\frac{U_1}{R_c} + I_1\left(\frac{R_s}{R_c} + 1\right)$$
$$U_2 = (U_1 - I_1 R_s) + \left[\frac{U_1}{R_c} - I_1\left(1 + \frac{R_s}{R_c}\right)\right] R_s$$
$$= U_1\left[1 + \frac{R_s}{R_c}\right] - I_1\left[R_s + R_s\left(1 + \frac{R_s}{R_c}\right)\right]$$
$$= U_1\left[1 + \frac{R_s}{R_c}\right] - I_1 R_s\left[2 + \frac{R_s}{R_c}\right].$$

Oct. 10, '41 (change signs above)

Want solutions where ratio of current to potential is the same at each end; so as to simplify calculation of effect due to whole series of filters.

Want $\frac{U_2}{I_2} = \frac{U_1}{I_1}$; call these characteristic sol'ns.

then $\frac{U_2}{U_1} = \frac{I_2}{I_1} = \lambda$.

Put in eq'ns:
$$\begin{cases} I_2 = -\frac{U_1}{R_c} + I_1\left(\frac{R_s}{R_c} + 1\right) = \lambda I_1 \\ U_2 = U_1\left(1 + \frac{R_s}{R_c}\right) - I_1 R_s\left(2 + \frac{R_s}{R_c}\right) = \lambda U_1 \end{cases}$$

or:
$$\begin{cases} \left(1 + \frac{R_s}{R_c} - \lambda\right) I_1 = \frac{U_1}{R_c} \\ \left(1 + \frac{R_s}{R_c} - \lambda\right) U_1 = I_1 R_s\left(2 + \frac{R_s}{R_c}\right) \end{cases}$$

For solutions, Det. of coeff. must vanish:

$$\left(1+\frac{R_s}{R_c}-\lambda\right)^2 = \frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right)$$

$$\therefore \lambda = 1+\frac{R_s}{R_c} \pm \sqrt{\frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right)}$$

Product $\lambda_1\lambda_2 = \left(1+\frac{R_s}{R_c}\right)^2 - \frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right) = 1$

Hence one λ gives increase in amplitude.
other λ " decrease " "

Set $\cosh^2 g = \left(1+\frac{R_s}{R_c}\right)^2$

$\sinh^2 g = \frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right)$.

then $\lambda = \cosh g \pm \sinh g$.

$$\therefore \begin{cases} \lambda_1 = e^{g} \\ \lambda_2 = e^{-g} \end{cases}$$

Now $\cosh g = 1+\frac{R_s}{R_c} = 1+2\sinh^2 g/2$.

$$\therefore \underline{\sinh \frac{g}{2} = \sqrt{\frac{R_s}{2R_c}}}$$

Now find $\frac{e_2}{I_2} = \frac{e_1}{I_1}$:

$$\frac{e_1}{I_1} = \mp Z = R_c\left(1+\frac{R_s}{R_c}-\lambda\right)$$

$$= R_c(\cosh g - \lambda)$$

$$= R_c\left(\cosh g - e^{\mp g}\right)$$

$$= \mp R_c \sinh g$$

Z called wave resistance

$Z = R_c \sinh g$.

Suppose we have a long line of filters; $\longrightarrow$ at end of

then U_n = pot. in n^{th} element

I_n = current " " "

and $U_n = -aZe^{n\gamma} + bZe^{-n\gamma}$

$I_n = ae^{n\gamma} + be^{-n\gamma}$

Suppose line ends with working impedance R_e;

then $U_n = R_e I_n$;

or: $-aZe^{n\gamma} + bZe^{-n\gamma} = aR_e e^{n\gamma} + bR_e e^{-n\gamma}$

$a(-Z - R_e)e^{n\gamma} = b(R_e - Z)e^{-n\gamma}$

or $\dfrac{a}{b} = \dfrac{Z - R_e}{Z + R_e} e^{-2n\gamma}$

if large n, R.P. $\gamma > 0$, and $Z \neq R_e$,

then $\frac{a}{b} \ll 0$

i.e. a decrease in current & E.M.F.

this determines b: $U_0 = bZ$

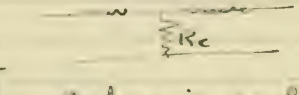
$\therefore b = \frac{U_0}{Z}$.

For $Z + R_e \doteq 0$: get resonance;

get sizable a, hence possibly amplification.

Some Possible filters:

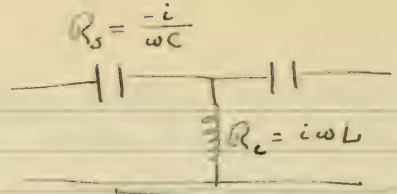
1): Pure resistances



then $\sinh \gamma/2 = \sqrt{\frac{R_s}{2R_c}}$ and γ is purely real; and $\neq 0$.

Same damping for any frequency; hence no good for wave filter.

2)



$$\sinh \gamma/2 = \sqrt{\frac{R_s}{2R_L}} = \sqrt{\frac{-i/\omega C}{2i\omega L}} = \sqrt{\frac{1}{-2\omega^2 LC}} = -i\sqrt{\frac{1}{2\omega^2 LC}}$$

Let $\frac{1}{2CL} = \omega_1^2$

then $\sinh \gamma/2 = -i\,\frac{\omega_1}{\omega}$.

$$\gamma/2 = b + ia$$

$$\sinh(b+ia) = -i\,\frac{\omega_1}{\omega}$$

$$\sinh b \cos a + i \cosh b \sin a = -i\,\frac{\omega_1}{\omega}$$

$$\therefore \sinh b \cos a = 0$$

$$\therefore \sinh b = 0 \quad \text{or} \quad \cos a = 0$$

$$b = 0 \quad \text{or} \quad a = \pm \pi/2.$$

also $\cosh b \sin a = -\frac{\omega_1}{\omega}$.

if $\frac{\omega_1}{\omega} < 1$, must take $b = 0$, $\cosh b = 1$

$$\therefore \sin a = -\frac{\omega_1}{\omega}$$

if $\frac{\omega_1}{\omega} > 1$ must take $a = \pm \pi/2$, $\sin a = \pm 1$

$$\therefore \cosh b = \frac{\omega_1}{\omega}$$

i.e. $\omega_1 > \omega$: $a = -\pi/2$
$b = \cosh^{-1}\frac{\omega_1}{\omega}$, $b > 0$ } here get damping

$\omega_1 < \omega$: $b = 0$
$a = \sin^{-1}\left(-\frac{\omega_1}{\omega}\right)$, $0 < a < -\pi/2$ } no damping.

$(\gamma = 2b + i\,2a)$. Transmits short waves
absorbs long waves }.

Oct. 13, '41 $\quad I_2 = I_1 e^{-g}$ (if g has a positive real part) (we showed other part $I_1' e^{+g}$ small for large g)

If real part of g vanishes, then the solutions $I_1 e^{-g}$ & $I_1' e^{+g}$ are equally likely — $\therefore\ I_2 = I_1 e^{-g} + I_1' e^{+g}$

Hence for $\omega > \omega_1$, we have $I_2 = I_1 e^{-2ia}$

$I_2' = I_1' e^{+2ia}$

Hence only a phase shift of $2a$.

For: $\omega = \infty$, ω_1

$a = 0$, $\pi/2$

Phase shift $= 0$, π (→ i.e. must operate in minimum elements)

Ampl. ratio: 1, 1

For $\omega < \omega_1$ $\quad a = -\pi/2$, phase shift $= -\pi$.

~~For ω~~ $\quad \cosh b = \frac{\omega_1}{\omega} = \frac{1}{2}(e^b + e^{-b})$

Let $x = e^b$:

$$x \frac{\omega_1}{\omega} = \frac{1}{2}(x^2 + 1)$$

$$\therefore\ x = \frac{\omega_1}{\omega} \pm \sqrt{\left(\frac{\omega_1}{\omega}\right)^2 - 1}$$

$$\therefore\ e^g = -e^{2b} \approx \left(\frac{2\omega_1}{\omega}\right)^2$$

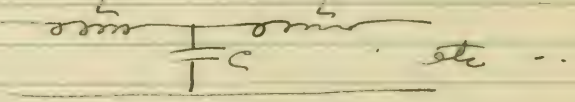
$\omega_1 \gg \omega$.

Hence for low frequencies, Amplitude ratio in successive elements is $\left(\frac{2\omega_1}{\omega}\right)^2$; very strong damping.

That filter not too good; only damps long waves — very long ones at that.

Suppose we want to damp short waves:

Take:



then $R_L = i\omega L$

$R_C = -\frac{i}{\omega C}$

since $\gamma_2 = \sqrt{\cancel{}\,i^2\omega^2 LC} = i\frac{\omega}{\omega_1}$.

$\omega_1 = \sqrt{\frac{1}{LC}}$ . .

Just the reverse of previous case;

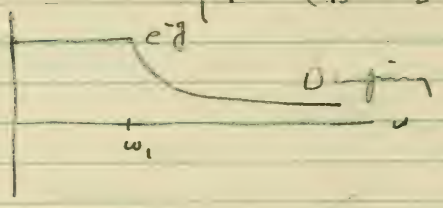
$\therefore$ $\omega > \omega_1$ gives damping

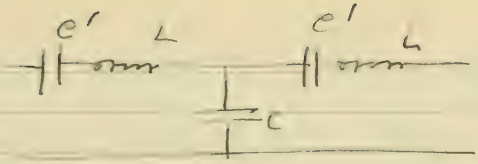
$\omega < \omega_1$ gives transmission.

For $\omega < \omega_1$: $\sin a = \omega/\omega_1$; $b = 0$

For $\omega > \omega_1$: $\cosh b = \omega/\omega_1$; $a = \pi/2$ (phase shift π)

$$e^{\gamma} = \left[\frac{\omega}{\omega_1} + \sqrt{\left(\frac{\omega}{\omega_1}\right)^2 - 1}\right]^2 \approx \left(\frac{2\omega}{\omega_1}\right)^2 \text{ for large } \omega \gg \omega_1$$





then $R_s = i\omega L + \frac{1}{i\omega C'}$

$R_c = \frac{1}{i\omega C}$

then $\sinh \gamma/2 = \sqrt{\frac{R_s}{2R_c}} = \sqrt{\frac{1}{2}i\left(\omega L - \frac{1}{\omega C'}\right) i\omega C}$

$= \sqrt{\frac{\omega^2 LC}{2} + \frac{C}{2C'}}$

For small ω, $\frac{R_s}{R_c} \approx \frac{C}{C'}$

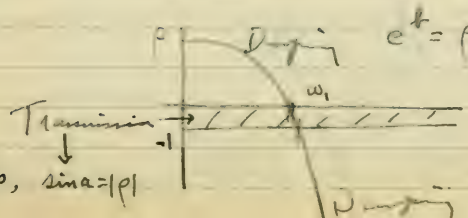
then $\sinh \gamma/2 = \sqrt{\frac{C}{2C'}}$ can take $C \gg C'$;

where $\omega = 0$, $\sinh \gamma/2 = \sqrt{\frac{C}{2C'}}$ $\therefore$ large damping constant while $\frac{C}{C'} \gg$

write $\frac{1}{C'L} = \omega_1^2$

then $\sinh \gamma/2 = \sqrt{\frac{C}{2C'}\left(1 - \frac{\omega^2}{\omega_1^2}\right)}$

hence as $\omega \to \omega_1$, $\sinh \gamma/2 \to 0$; come into region of transmission; then $\sinh \gamma/2$ becomes $< \sqrt{-1}$, get damping again.



$e^{\gamma} = \rho + \sqrt{\rho^2 + 1}$, $a = 0$

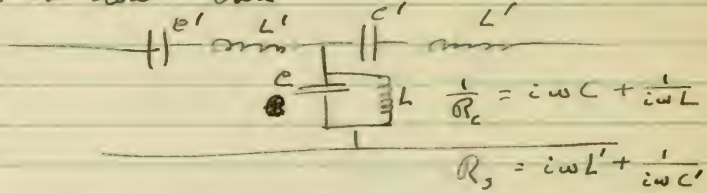
$\frac{R_s}{2R_c} = \rho$

$e^{\gamma} = |\rho| + \sqrt{\rho^2 - 1}$, $2a = \pi$

this has drawback that damping is constant at low ω.

Would like a filter that transmits a narrow band like this one, but damps low frequencies more effectively than this one.

Can be done thus:



this will give $\to \infty$ damping for both low + high freq.

$$\rho = \frac{R_s}{2R_c} = \frac{1}{2}\left(i\omega C + \frac{1}{i\omega L}\right)\left(i\omega L' + \frac{1}{i\omega C'}\right)$$

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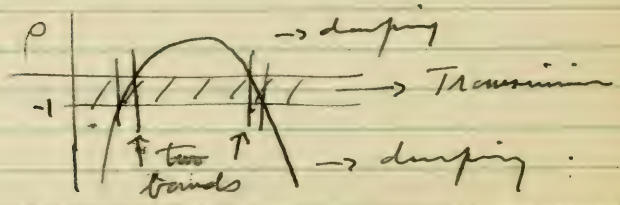
$$= \frac{1}{2}\left(-\omega^2 CL' - \frac{1}{\omega^2 LC'} + \frac{C}{C'} + \frac{L'}{L}\right)$$

$\therefore \rho \to -\infty$ for $\omega \to$ either ∞ or 0.

$\therefore$ large damping for both lo + hi freq.

$$\rho = -\frac{1}{2}\cdot\frac{1}{C'L\omega^2}\left[(1-L'C'\omega^2)(1-LC\omega^2)\right]$$

If $L'C' > LC$, one factor is (−) while other is (+); Hence ρ is positive.



Desirable for ρ not to be positive, so as to get

only one band; this can be done by making $L'C' = LC$.

then $\rho = -\frac{1}{C'L\omega^2}\left(1-\frac{\omega^2}{\omega_0^2}\right)^2$ is always negative.

then transmission band is in vicinity of the resonant freq. $\omega_0^2 = \frac{1}{LC} = \frac{1}{L'C'}$;

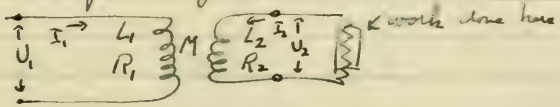
then $\sin\psi_2 = \frac{i}{\sqrt{2C'L\omega^2}}\left(1-\frac{\omega^2}{\omega_0^2}\right)$.

$$= \frac{i}{\sqrt{2\frac{L'}{L}}}\frac{\omega_0}{\omega}\left(1-\frac{\omega^2}{\omega_0^2}\right)$$

$$= i\sqrt{\frac{2L'}{L}}\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right).$$

For sharp resonance, i.e. narrow band transmitt^{n}, make $L' >> L$.

Back to transformer problem:



$$U_1 = I_1(R_1+i\omega L_1) + i\omega M I_2$$

$$U_2 = I_2(R_2+i\omega L_2) + i\omega M I_1$$

$$I_2 = \frac{U_1 - I_1(R_1+i\omega L_1)}{i\omega M}.$$

$$U_2 = \frac{U_1 - I_1(R_1+i\omega L_1)}{i\omega M}(R_2+i\omega L_2) + i\omega M I_1$$

$$= U_1\frac{R_2+i\omega L_2}{i\omega M} - \frac{(R_1+i\omega L_1)(R_2+i\omega L_2)+\omega^2M^2}{i\omega M}I_1$$

~~look for sol'n~~ : $\cancel{= \lambda U_1}$.

Suppose now R_1, R_2 are small, and coupling $k \cong 1$.
then second term of U_2 vanishes;

and $$\frac{U_2}{U_1} = \frac{L_2}{M} = \sqrt{\frac{L_2}{L_1}} = \frac{n_2}{n_1}.$$

$$I_2 = \frac{U_1}{i\omega\sqrt{L_1 L_2}} - I_1\sqrt{\frac{L_1}{L_2}}.$$

if $U_1 \ll I_1 L_1 \omega$,

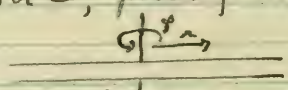
$$-\frac{I_2}{I_1} = \sqrt{\frac{L_1}{L_2}} = \frac{n_1}{n_2}.$$

Justified if $n_2 \gg n_1$.

---

ublem: t other lution

So far our circuits have been different from mechanical cases, in that we have always gotten just one (1 unit) 2 (2 unit) etc. characteristic frequencies; nothing like resonance to overtones or harmonics — this is because C, L, R are independent of ω. Actually both C & R depend considerably on ω; For R, known as "skin effect" — For C, potential is no longer constant over surface:



disc condenser.

Field intensity E_z - ordinarily the same throughout condenser. It has a time derivative:

$$\frac{dE_z}{dt} \to c \operatorname{curl} H \qquad \text{(Maxwell)}$$

~~ex~~ H will be in φ direction H_φ, a function of r. For low ω: $H_\varphi \cdot 2\pi r = \int c\, \dot{D}_z$
$= c\varepsilon i\omega E_z \pi r^2$

$\therefore \quad H_\varphi = i\omega c \varepsilon \frac{1}{2} E_z$

But $\frac{\partial H_\varphi}{\partial t}$ will produce a field equal $\underline{E}$, in φ direction; causes $\frac{\partial E_z}{\partial r} \sim \omega^2 E_z$.

Hence r-dependence of E_z will increase with ω^2.
For low frequency, can neglect r-dependence; quasi-stationary state; no longer true for rapidly oscillating field.
$\therefore$ For large ω: must have $E_z(r)$ $H_\varphi(r)$.

Simplest case of resonating cavity; Useful for generation of ultra-high freq. waves.

Oct. 17, '41 [Digression on 1st problem: In calculating L, theoretical formula is $L = \frac{4\pi^2 a^2 n^2}{b}$ a = radius, b = length, n = no. of turns.
Derived by ~~[illegible]~~ integ. of H; assume no field outside - post if of finite length. Correction $K\left(\frac{b}{2a}\right)$ can be found tabulated - (is a complete elliptic integral). Formula without correction in our problem gives 24.7 μh. K = .90, whence L = 22.3 μH. Further corrections due to finite thickness; if c is depth of winding, substract $4\pi \frac{n^2 a c}{b}\left(\log 2 + B\left(\frac{b}{a}\right)\right)$
this gives 3.9 μH; $\therefore$ L = 18.4 μh.
Ham handbook gives: $L = \frac{.2 A^2 n^2}{3A + 9B + 10C}$ A = diam. winding, B, depth, and C, thickness, inches.
= 18.7 μh in this case.]

Problem 4: Want to transmit 1,000 kc: in fact want to transmit a band 1,000 ± 10 kc: 1) Find filter constants. [Small mallort C: 100 cm.] 2) What is resistance of wire; - (again 1 mm Cu wire - 1 mm diameter of coil (simple formula for L no corrections) 3) Calculate performance at resonance. 4) 950 kc is to be damped out to 10^{-5} of signal strength; $\therefore$ how many filters.

At low ω, some E thru out condenser at a given time;

But for high ω, E takes some time to be propagated from entry where it is first found, to rim.

Assume circular condenser; then $E_z = E_z(r, \varphi)$.
[Here E_z will not depend on φ]

Will use wave equation [2nd order eq'n from Maxwell's eq'ns]

$$\nabla \times \underline{H} = \frac{\epsilon}{c}\dot{\underline{E}} \qquad \epsilon = \text{dielectric const.}$$
$$\nabla \times \underline{E} = -\frac{\mu}{c}\dot{\underline{B}} \qquad \mu = \text{permeability.}$$
$$= -\frac{\mu}{c}\dot{\underline{H}} \qquad \mu \cong 1.$$

Whence $\nabla^2 \underline{E} = \frac{\epsilon}{c^2}\ddot{\underline{E}}$ (div $\underline{E} = 0$ — no space charge)

gives waves of velocity $\frac{c}{\sqrt{\epsilon}}$

where $\sqrt{\epsilon} = n$ (refraction index).

Here $E_z = E_0 e^{i\omega t}$

$\ddot{E}_z = -\omega^2 E_z$; in cylindrical coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

whence $\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} = -\frac{\omega^2 \epsilon}{c^2} E$.

Can see immediately that $E \sim e^{im\varphi}$

Whence $\frac{d^2 E}{dr^2} + \frac{1}{r}\frac{dE}{dr} - \frac{m^2}{r^2}E + k^2 E = 0$

$$k = \frac{\omega}{v} = \frac{\omega}{c/\sqrt{\epsilon}} = \sqrt{\epsilon}\,\frac{\omega}{c}.$$

(k = no of waves in 2π cms.)

"This is equation of Bessel: order m, argument kr."

Hence $E \sim J_m(kr)$. (other Bessel

f'n blow up at origin, hence take only J)

Whence $E_z = a_m J_m(kr)\, e^{im\varphi} e^{i\omega t}$.

(In general a complex).

Now $J_m(kr) \sim (kr)^m$ for small r. ; Hence for $m=0$, $E=0$ where lead comes in! ; also, φ dependence requires $E=0$ at center.

For Lead at center, $m=0$,

and $E_z = a\, J_0(kr)\, e^{i\omega t}$.

Oct. 20

Based F's reference: Stratton, E.M. Theory

Want to fid charge on condenser.

Charge density at surface $= \frac{\varepsilon E}{4\pi}$;

Total charge $= \int_0^R \frac{\varepsilon E}{4\pi} \cdot 2\pi r\, dr$

$= \frac{\varepsilon}{2} \int_0^R E\, r\, dr$

$= \frac{\varepsilon a}{2} \int_0^R J_0(kr)\, r\, dr$

For k very small, get static case;

$J_0(kr) = 1$

$Q = \text{Total charge} = \frac{\varepsilon a}{2} \cdot \frac{R^2}{2}$

For k not very small:

$Q = \frac{\varepsilon a}{2} \frac{1}{k^2} \int_0^{kR} J_0(x)\, x\, dx$

$= \frac{\varepsilon a}{2k^2} \cdot kR J_1(kR)$

$= \frac{\varepsilon a}{2k} R J_1(kR)$.

$\therefore$ Ratio of capacity $k=l$ to cap. $k=0$,

is : $\dfrac{C_k}{C_0} = \dfrac{2}{kR} J_1(kR)$.

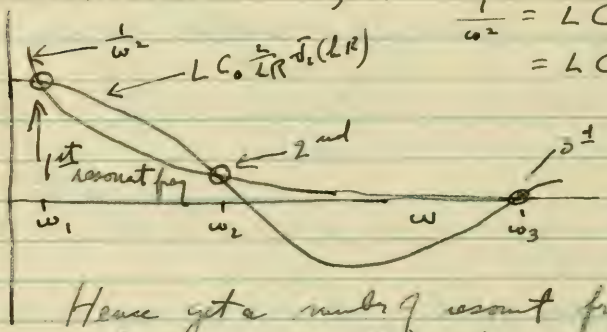
(small k, $J_1(kR) \to \frac{kR}{2}$, and $\frac{C_l}{C_0} \to 1$).

In general, $\frac{C_l}{C_0} < 1$.

For nodes of $J_1(kR)$, C_k vanishes.

Consider circuit containing this capacity & an inductance : L, C.

$$\frac{1}{\omega^2} = LC$$
$$= L C_0 \frac{2}{kR} J_1(kR).$$



(assuming L indep. of ω)

Hence get a number of resonant freq. for a given condenser!

Except for 1st, Charact. freq. are very nearly at the nodes of $J_1(kR)$.

Zeros of J_1 : First zero ~ 4 Later n, near $\pi(n+\frac{1}{4})$

$kR \approx \pi(n+\frac{1}{4})$

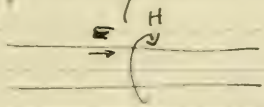
$\lambda = \frac{2\pi}{k} \approx \frac{2\pi R}{\pi(n+\frac{1}{4})} = \frac{2R}{n+\frac{1}{4}}$. approx. n wave lengths to diameter,

1st:

So we get overtones. - but no simple connection between 1st & others - 1st one given by static formula $\frac{1}{2c}$, others depend on dimensions of conductor -

Skin effect:
frequency dependence of resistance.

Conductor:



$$\nabla \times \underline{H} = \frac{4\pi}{c}\underline{j} + \frac{\epsilon}{c}\dot{\underline{E}}$$

$$\nabla \times \underline{E} = -\frac{\mu}{c}\dot{H}$$

Neglect $\dot{\underline{E}}$ in comp. with $\underline{j}$; $\underline{j} = \sigma \underline{E}$

Then $\nabla \times \nabla \times E = -\frac{4\pi\mu}{c^2}\sigma \dot{E}$

if again $\nabla \cdot E = 0$: $\nabla^2 E = \frac{4\pi\mu\sigma}{c^2}\dot{E}$

if $E \sim E e^{i\omega t}$

then $\nabla^2 E = \frac{4\pi\mu\sigma i\omega}{c^2} E$.

or $\nabla^2 E = -k^2 E$

$\sigma_{e.s.u}$: $\left[\frac{j}{E}\right]$

$= \sigma_{e.m.u.} \cdot c^2$

$= \frac{c^2}{10^9}\sigma_{mho/m}$

k^2 is pure imaginary:

$k^2 = -\frac{2\pi\mu\sigma_{mho}}{10^9} \cdot \omega \cdot 2i$

$k = \sqrt{\frac{2\pi\mu\sigma}{10^9}\omega}\,(1+i)$

Symmetrical solution is again:
(solid wire)

$E = a\, J_0(kr)$.

For $kR \ll 1$, $J_0 = 1$, & E is ~~constant~~ ~~throughout~~ the wire; hence j constant ~~throughout~~ wire.

(The coefficients all start, hence must be in sign —)

Otherwise J_0 increases with increasing absolute value of complex argument. $J_0(x) = \frac{1}{2}\sqrt{\frac{2}{\pi x}}\left[e^{i(x-\pi/4)} + e^{-i(x-\pi/4)}\right]$.

$x \gg 21 \rightarrow$

$k = k_0(1+i)$

$x = k_0 R(1+i)$ $\qquad e^{-i(x-\pi/4)} \rightarrow e^{-i(k_0R(1+i)-\pi/4)}$

$= e^{k_0 R}$

hence increases.

$\therefore$ j also is larger near surface — in fact practically all current at surface — so called skin effect.

F.T 22,41 For Cu: $\sigma \cong 6\cdot 10^5$ mho/cm. $\mu = 1$ $\omega = 2\pi f$

$$k_0 = \sqrt{\frac{2\pi\, 5\mu\omega}{10^9}}$$

$$= 2\pi\sqrt{\frac{6\cdot 10^5\cdot 6}{10^9}}$$

$$\cong 2\pi\cdot 10^{-2}\sqrt{6f}$$

$$\cong .15\sqrt{f} \text{ cm}^{-1}.$$

For A.C., $f = 60$, $k_0 \approx 1$.

$\therefore k_0 r \sim 1$ for $r = 1$ cm.

i.e. ordinary A.C. will start to show skin effect for Cu conductors ~ 1 cm in radius.

For $f \sim 10^6$ $k_0 \sim 150$ cm^{-1}

Now even wire of .1 diameter ($r = .05$ cm) $k_0 r = 7.5$ — large skin effect.

What to do about it? Can use a hollow conductor. But only for large diameters —

Or take large number of small wires ∴ But then int
such important – especially if many, closely packed —
Might arrange small wires in hollowing thus : ∴ !

Total current : $I = \int_0^{r_0} j \, 2\pi r \, dr$

Compare with $I_1 = j(r_0) \cdot \pi r_0^2$ current at surface of wire

$$\frac{I}{I_1} = \frac{R_\omega}{R_0} \frac{R_0}{R_\omega}$$

$j = J_0(kr)$ $\quad \therefore I = 2\pi \int_0^{r_0} J_0(kr) \, r \, dr$

$$= \frac{2\pi}{k} r_0 J_1(kr_0)$$

$$\therefore \frac{I}{I_1} = \frac{\frac{2\pi}{k} r_0 J_1(kr_0)}{\pi r_0^2 J_0(kr_0)}$$

$$= \frac{2}{kr_0} \frac{J_1(kr_0)}{J_0(kr_0)} = \frac{R_{\omega 0}}{R_{0\omega}} .$$

a) if $|kr_0| << 1$: $J_1 = \frac{1}{2} k r_0$

$J_0 = 1$

$\therefore R_\omega \to R_0$.

Or expand further & get idea when effect starts.

b) $|kr_0| >> 1$:

$$J_1 = \frac{1}{2}\sqrt{\frac{2}{\pi x}} \, e^{i x - i \frac{3}{4}\pi}$$

$$J_0 = \frac{1}{2}\sqrt{\frac{2}{\pi x}} \, e^{i x - i \frac{\pi}{4}}$$

$$\frac{2 J_1}{J_0} = \frac{2}{x} i$$

$$x = k_0 (1 \mp i) r_0$$

$$\frac{R_\omega}{R_0} \to \frac{k_0 r_0}{1 \mp i} = \frac{1}{2} k_0 r_0 (1 + i)$$

$$\left|\frac{R_\omega}{R_0}\right| >> 1 .$$

Also R_ω is now complex : effective inductance equal in magnitude to resistance.
[using previous example, $\left|\frac{i\omega}{1-i\omega\tau}\right| = \frac{1}{2}\omega\tau_0 \sim 1$]

$$R_\omega = \frac{1}{2} R_0 k_0 l_0$$

$$R_0 \sim \frac{1}{\sigma l_0^2} \quad , \quad k_0 \sim \sqrt{\sigma}$$

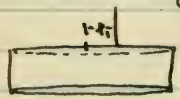
$$\therefore R_\omega \sim \frac{\sqrt{\sigma}\, l_0}{\sigma l_0^2} \sim \frac{1}{\sqrt{\sigma}\, l_0}$$

$\therefore$ R_ω still depends on σ & l_0 in same way as R_0, but only as square root —

All this neglected term $-\frac{\mu\varepsilon\omega^2}{c^2}E$ in wave eq'n this will start to come in at $\omega \gtrsim 10^{19}$ — because $\varepsilon \sim -10^5$ in metals on account of free electrons.

Limiting skin effect case (thick conductor or high freq) can be treated simply, without Bessel f'n's.

Set. 24 problem. Cavity of cylindrical shape:
Excite at distance r_1 from center:



Non symmetrical oscillation
1) What is frequency?
2) If not closed, what is effective capacity?

$k \sim \sqrt{\sigma\mu\omega}$ For ferromagnetics, thinner skin.
$\underline{B} = \mu \underline{H}$ usually holds for $B < 10^4$ gauss.

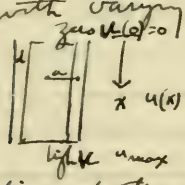
For long straight wire, $H = \frac{.2\, I_{amp}}{r}$ if $r \ll r_{max}$,

$H = 2I$ amp. $\therefore$ $B = 2\mu$ for $I = 1$ amp.
So for most materials formula is valid, not for very high permeabilities, e.g. permalloy.

Coil at high freq.
Main disturbing effect is distributed capacity.
Hard to calculate; not too hard to estimate.

Consider coil of two layers; (in + out at same end)

Greatest pot. diff at end;
We have a sort of cylindrical condenser, with varying pot. diff.



(radius $\gg$ separation)

Cap./unit length $= C = \frac{2\pi a}{4\pi d} = \frac{a}{2d}$.

Q = charge/cm $= C\,u(x)$

$$\frac{\partial Q}{\partial t} = -\operatorname{div} I = -\frac{\partial I}{\partial x} = C\frac{\partial u}{\partial t}.$$

$$\text{Field} = -\frac{\partial u}{\partial x} = R\,I + L\frac{dI}{dt}.$$

R = ohms/cm.
L = henry/cm.

Eliminate + get:

$$\frac{\partial^2 u}{\partial x^2} = -R\frac{\partial I}{\partial x} + LC\frac{\partial^2 u}{\partial t^2}.$$

$$= RC\frac{\partial u}{\partial t} + LC\frac{\partial^2 u}{\partial t^2}.$$

If R is small; then

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2}$$

$$u \sim e^{i\omega t}\sin(\sqrt{LC}\,\omega x)$$

$u(0)=0$ $\qquad I \sim \cos(\sqrt{LC}\,\omega x)$

∴ Current in coil is smaller (for short coil) at leads than at middle & top! (round up in charging condenser) If length of coil is b, then for $\sqrt{LC}\,\omega b \simeq \pi/2$, no current at all!

Natural frequency: $\omega = \dfrac{\pi}{2b\sqrt{LC}}$.

L, C are cap + ind. / unit length;

$\therefore \quad \omega = \dfrac{\pi}{2\sqrt{L'C'}}$ $\quad L'$, C', total ind., cap.

Estimate: $L' = \dfrac{4\pi^2 a^2 n^2}{b}$ $\quad C' = \dfrac{ab}{2d\,c^2}$.

$$\sqrt{L'C'} = \sqrt{\frac{4\pi^2 a^2 n^2 a b}{2b\,d\,c^2}}$$

$$= \frac{2\pi a n}{c}\sqrt{\frac{a}{2d}}.$$

$$\lambda_{crit.} = \frac{2\pi c}{\omega} = \frac{2\pi c \cdot 2\sqrt{L'C'}}{\pi}$$

$$= 4c \cdot \frac{2\pi a n}{c}\sqrt{\frac{a}{2d}}$$

$$= 4\sqrt{2}\,\pi a n\sqrt{\frac{a}{d}}.$$

$$= 2\pi a n\sqrt{\frac{8a}{d}}.$$

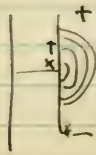
$2\pi a n \approx$ length of wire in coil.

∴ make length of wire small comp. with desired wave length.

Side ~~[illegible]~~ coil: $q = 2\pi a \cdot$ charge/unit area (charge/cm.)

$= 2\pi a \cdot \frac{1}{4\pi} \cdot E$ $\quad E = \dfrac{2u}{\pi x}$!

$= \dfrac{a}{2} \cdot \dfrac{2u}{\pi x}$ $\quad$ pot. diff divided by length of field lines!



lines of force.

again $\left\{ -\dfrac{\partial I}{\partial x} = \dfrac{a}{\pi x}\dfrac{\partial u}{\partial t} \right.$

$\left\{ -\dfrac{\partial u}{\partial x} = L\dfrac{\partial I}{\partial t} \right.$ if only inductance

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{ab}{\pi x}\frac{\partial^2 u}{\partial t^2}$$

Or, in consistent units: $\frac{\partial^2 u}{\partial x^2} = \frac{ab}{\pi x c^2}\frac{\partial^2 u}{\partial t^2}$.

$$u \sim \sqrt{x}\, J_1\left(\sqrt{\frac{4}{\pi} L a x}\ \frac{\omega}{c}\right)$$

$$\lambda_a \sim \sqrt{L a} \qquad L' = Lb.$$

Oct. 27 $\quad J_1(y) = \frac{1}{2} y$ for small y.

$\therefore$ for small x, $u \sim \sqrt{x} \cdot \frac{1}{2}\sqrt{\frac{4}{\pi} L a x}\,\frac{\omega}{c}$

$\sim x \cdot \sqrt{K}$.

Let $K = \sqrt{\frac{L a \omega^2}{\pi c^2}}$

For large ω $\quad \frac{J_1}{\frac{1}{2} y} < 1$

Get I from second eq'n:

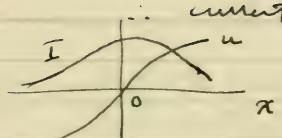
$$\frac{\partial u}{\partial x} = -iL\omega I = \frac{1}{\sqrt{2\sqrt{K}}}\frac{d}{dx}\left(y J_1(y)\right)$$

$$y = \sqrt{2Kx}$$

$$\frac{1}{2K}\frac{d}{dx} = \frac{1}{2y}\frac{d}{dy}$$

$$\therefore -Li\omega I = \sqrt{\frac{K}{2}}\, J_0(y).$$

$\therefore$ current is again distributed like J_0



[in double layer case, we found $u \sim \sin x$, $I \sim \cos x$, same general appearance].

At first node of J_0: $y = 2.8 = 2\sqrt{Kx}$

if length of coil is b, then u vanishes for

$$K b/2 = \frac{(2.8)^2}{4} = 1.4$$

$$K = \frac{2 \cdot 2.8}{4b} = \frac{1.4}{b} = \frac{2.8}{b}.$$

Whence $\frac{\omega_{crit}^2}{c^2} = \frac{2 \cdot 8 \cdot \pi}{Lab}$

$$\therefore \frac{\omega_{crit}}{c} = \sqrt{\frac{28\pi}{Lab}}$$

and $\lambda_{crit} = \frac{2\pi c}{\omega_{crit}} \simeq 2\sqrt{Lab}$

$= 2\sqrt{L'}\, a$ $\qquad$ L' = total inductance

$= 2\sqrt{\frac{4\pi^2 n^2 a^2}{b} \cdot a}$

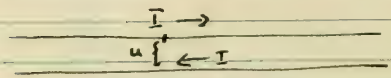
$= 2\pi n a \cdot \sqrt{4\frac{a}{b}}$ $\qquad$ $b \gg a$ probably ?

$2\pi n a$ = length of wire;

For double layer, had $\lambda_{crit} = 2\pi n a \sqrt{8\frac{a}{d}}$ where $d \ll a$.

Hence λ_{crit} for double layer much greater than for single layer.
$\therefore$ single layer coil better at high frequencies.

PROPAGATION OF 'ELECTRIC' WAVES ALONG WIRES.



Two wires; at each pt. a diff. of pot. u between wires, current I in wires.

Even at low freq., current not the same everywhere.
Both u & I are functions of both x & t.

Consider a unit of length; u changes by $\frac{du}{dx}$; this is accounted for by $RI + L\frac{dI}{dt}$: $\quad -\frac{du}{dx} = RI + L\frac{\partial I}{\partial t}$ $\qquad$ R, L, per unit length.

$$-\frac{\partial I}{\partial x} = \frac{\partial q}{\partial t} = C\frac{\partial u}{\partial t} + Gu \qquad C \text{ "}$$

G accounts for leaks, corona, etc. $\leftarrow$ $\qquad$ to make it symmetrical

Oct. 29, 1941 Equations of transmission line:

$$\begin{cases} -\dfrac{\partial I}{\partial x} = C\dfrac{\partial u}{\partial t} + G u \\ -\dfrac{\partial u}{\partial x} = L\dfrac{\partial I}{\partial t} + R I \end{cases}$$

all quantities per unit length; usually convenient to take a large unit, as e.g. km.

Parallel wires: $C = l_1 \frac{1}{4}\left(\ln \frac{a}{r}\right)^{-1}$ $\quad l_1$ = length of unit (e.g. 10^5)

$L = 4 l_1 \left(\ln \frac{a}{r}\right)$

r = radius of wire

a = distance between wires.

Concentric cable: $C = l_1 \left(2 \ln \frac{r_1}{r_2}\right)^{-1}$

$L = l_1 \cdot 2 \ln \frac{r_1}{r_2}$

$C > l_1$, $L < l_1$ for concentric cable
$C < l_1$, $L > l_1$ " parallel wires } roughly.

Hence per km.: $C \sim 10^5$ cm $= 0.1\ \mu F$

$L \sim 10^5$ cm $= 0.1$ mH.

for Cu: $R = 1.7 \cdot 10^{-6} \cdot \frac{10^5}{.031} \sim 5$ ohm/km

if $d \sim .2$ cm. $A_{cm} = \pi \cdot .01$

$G \sim 10^{-6}$ mho/km (Villing)

[If G is to be this, then conductivity of insulating material should be $\leq 10^{-11}$ mho/cm].

Assume $u = u(x)\, e^{i\omega t}$

$I = I(x)\, e^{i\omega t}$

then $-\dfrac{\partial I}{\partial x} = u(G + iC\omega)$

$-\dfrac{\partial u}{\partial x} = I(R + iL\omega)$

Combining : $\frac{\partial^2 u}{\partial x^2} = u(R+iL\omega)(G+iC\omega)$

$$\therefore \quad u \sim e^{kx}$$

where $k^2 = (R+iL\omega)(G+iC\omega)$

k is wave no. in km^{-1} ; x measured in km.

$$k^2 = (GR - CL\omega^2) + i\omega(RC + GL)$$

$$k = i\alpha + \beta$$

$$-\alpha^2 + \beta^2 = -CL\omega^2 + GR$$

$$2\alpha\beta = \omega(RC + GL)$$

Want a line with $GR << CL\omega^2$;

hence $\beta << \alpha$

$$\alpha \approx \omega\sqrt{CL}$$

$$\beta \approx \frac{\omega(RC+GL)}{2\omega\sqrt{CL}} = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

$$\therefore \quad u_1 \sim e^{i\alpha x + \beta x} \quad \text{or} \quad e^{-i\alpha x - \beta x} = u_2 .$$

In general, $u = a_1 u_1 + a_2 u_2$

In long line, only exp. decaying sol'n will be important (except at end). Hence

$$u = e^{-i\alpha x - \beta x} \cdot e^{i\omega t}$$

$$= e^{i(\omega t - \alpha x)} \cdot e^{-\beta x}$$

damped travelling wave ; Wave velocity $= \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{CL}}$

$$CL \approx \frac{l_1^2}{c^2} ;$$

$\therefore$ Wave velocity $\approx \left(\frac{c}{l_1}\right)$ km/sec

In cable with dielectric : $v \sim \frac{c}{\sqrt{\epsilon}}$.

$\therefore$ Get running waves travelling with almost the velocity of light ; damped by factor $e^{-\beta x}$.

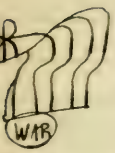
Damping per km : $\beta = \frac{1}{2}\left(\frac{R}{\sqrt{L/C}} + G\sqrt{L/C}\right)$

$$\sim \frac{1}{2}\left(\frac{5}{30} + 30 \cdot 10^{-6}\right) \sim \frac{1}{12} .$$

$R \sim 5$

$\sqrt{L/C} \sim 30$

$G \sim 10^{-6}$



Damping almost entirely due to ohmic resistance.

Wire:	Diam. wire:	G	L (mh/km)	C (μF/km)	R
Cu.	1.5 mm	10^{-6}	2.3	.0051	31
	5 mm.	10^{-6}	1.8	.0068	1.9
Cable	.6 mm	10^{-6}	.7	.031	130
	2.0 mm	10^{-6}	.6	.041	12

More damping in cable — $\frac{R}{\sqrt{L/C}} \sim 1/3$

For most favorable case, can get $R/\sqrt{L/C} \sim \frac{1}{250}$

i.e. 500 km to drop to 1/e of value. 1/3 for

Oct 31.

	Diam.	removal.	$\frac{1}{2} R\sqrt{\frac{C}{L}}$ ← f^{-1} →	$\frac{1}{2} G\sqrt{L/C}$	Ratio	
Free wire	1.5 mm	2.93	.0157	.00033	50	62
	5 mm.	2.87	.00186	.00025	7.5	475
Cable	.6 "	2.15	.43	.00007	6,200	2.3
	2.0 "	2.02	.049	.00006	800	20

We have assumed above : $L\omega \gg R$ i.e. $\omega \gg \omega_1 = R/L$

$C\omega \gg G$ " $\omega \gg \omega_2 = G/C$.

For above cases, :

$\frac{\omega_1}{2\pi}$ (kc)	$\frac{\omega_2}{2\pi}$ (kc)
2.2	.03
.17	.025
30	.005
3.1	.004

∴ $f_1 = \frac{\omega_1}{2\pi}$ is about in region of telephone transmission.

Damping needs to be reduced — esp. for cables —

Damping is given by $R\sqrt{\frac{C}{L}}$; R can't be reduced; C can't be reduced - esp. if a cable - which is desirable — is used. Hence hope to increase L. Pupin — put in a coil from time to time to do this.

If we have Pupin coils of L_p and R_p placed distance s apart; Additional L is L_p/s, R_p/s

so that $\beta = \frac{1}{2}\left(R + \frac{R_p}{s}\right)\sqrt{\frac{C}{L + L_p/s}} + G\sqrt{\frac{L + L_p/s}{C}}$.

For low freq. Neglecting R_p/s, Minimum damping would be for two terms equal i.e. $R\sqrt{\frac{C}{L'}} = G\sqrt{\frac{L'}{C}}$

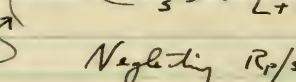
this gives $L' = \frac{RC}{G} = 4$ henry/km too much!

P. will also behave like a filter - critical frequency

$\omega_0 \sim \frac{1}{\sqrt{(L + L_p)Cs}}$ so that increasing s will lower ω_0 also decreasing L_p " " "

So there is another limitation on pupin coils —

theory:



P.d. decrease $\Delta u = -I R_p$ in coil

I in line $-\frac{\partial u}{\partial x} = (R + iL\omega) I$ $\qquad R_p = R_p + i\omega L_p$.

$-\frac{\partial I}{\partial x} = (G + iC\omega) u$

then $I = a e^{hx/s} + b e^{-hx/s}$

$h = s\sqrt{(R + iL\omega)(G + iC\omega)}$

If at $x=0$ $I=1$, $\quad a+b=1$.

$$u = -Z a e^{hx/s} + Z b e^{-hx/s} \qquad Z = \text{wave resistance}$$

for $u \sim e^{hx/s}$, $-\frac{\partial u}{\partial x} = -\frac{h}{s} e^{\frac{h}{s}x} = -\sqrt{\quad}\, u = (R+iL\omega)\, I$

$$\frac{u}{I} = -\sqrt{\frac{R+iL\omega}{G+iC\omega}} = -Z.$$

$\frac{u}{I}$ at left $= V$

at right of coil $= V - R_p$

$\therefore$ if $I=1$, get a condition:

$$Z(b-a) = V - R_p.$$

$$\therefore a = \frac{Z - V + R_p}{2Z}$$

$$b = \frac{Z + V - R_p}{2Z}.$$

at left end of next coil,

again $\frac{u}{I}(s) = V$

& so on $\quad \therefore$ for $x=s$

$$V = a = -Z a e^{h} + Z b e^{-h}$$

$$V(a e^{h} + b e^{-h}) = Z(-a e^{h} + b e^{-h})$$

Vor. 3

$$(Z-V)(Z+V-R_p)\, e^{-h} = (Z+V)(Z-V+R_p)\, e^{h}$$

$$(Z^2 - V^2)(e^{-h} - e^{h}) = R_p\left[Z(e^{h} + e^{-h}) + V(e^{h} - e^{-h})\right]$$

$$\therefore -Z^2 + V^2 - V R_p - Z R_p \coth h = 0$$

$$\therefore V = \frac{R_p}{2} \pm \sqrt{\tfrac{1}{4}R_p^2 + Z^2 + Z R_p \coth h}$$

if define Wave resistance : $\frac{u\,(\text{centre of coil})}{I\,(\quad " \quad)} = V - \frac{1}{2}R_p$.

$$= \pm\sqrt{\tfrac{1}{4}R_p^2 + Z^2 + Z R_p \coth h}$$

Want to know $\lambda = \frac{I(s)}{I(0)}$.

$$= a e^{h} + b e^{-h}$$

$$(Z-V)\, b e^{-h} = (Z+V)\, a e^{h}$$

$$\therefore \lambda = a e^{h} \frac{2Z}{Z-V} = e^{h} \frac{Z - V + R_p}{Z - V}.$$

Expect $\lambda_1\lambda_2 = 1$ $\quad$ ($\therefore$ $\lambda_1 = e^{g}$ $\lambda_2 = e^{-g}$).

$$\lambda_1\lambda_2 = e^{2h}\left(\cancel{1+\frac{R_p}{z-V_1}}\right)\left(\cancel{1+\frac{R_p}{z-V_2}}\right)$$

$$= e^{2h}\left[\frac{(z-V_1+R_p)(z-V_2+R_p)}{(z-V_1)(z-V_2)}\right]$$

But $V_1+V_2 = R_p$

$V_1V_2 = -z^2 - zR_p \coth h$ $\quad$ from quadratic giving

$$\therefore (z-V_1)(z-V_2) = z^2 - zR_p - z^2 - zR_p \coth h$$

$$= -zR_p\left[1+\coth h\right].$$

and $(z+R_p-V_1)(z+R_p-V_2)$

$$= (z+R_p)^2 - (z+R_p)R_p - z^2 - zR_p \coth h$$

$$= zR_p(1-\coth h).$$

$$\therefore \lambda_1\lambda_2 = e^{2h}\,\frac{zR_p(1-\coth h)}{-zR_p(1+\coth h)}$$

$$= -e^{2h}\left[\frac{e^{h}-e^{-h}-e^{h}-e^{-h}}{e^{h}-e^{-h}+e^{h}+e^{-h}}\right]$$

$$= -e^{2h}\cdot\frac{-2e^{-h}}{2e^{h}} = 1$$

$$\tfrac{1}{2}(\lambda_1+\lambda_2) = \cosh g = \tfrac{1}{2}e^{h}\left[1+\frac{R_p}{z-V_1}+1+\frac{R_p}{z-V_2}\right]$$

$$= e^{h}\left[1+\tfrac{1}{2}R_p\,\frac{2z-V_1-V_2}{(z-V_1)(z-V_2)}\right]$$

$$= e^{h}\left[1+\tfrac{1}{2}R_p\,\frac{2z-R_p}{-zR_p(1+\coth h)}\right]$$

$$= e^{h}\left[1 - \frac{R_p}{2}\cdot\frac{2Z - R_p \sinh h}{Z R_p(\cosh h + \sinh h)}\right]$$

$$= e^{h}\left[1 - \frac{1}{2}\,\frac{2Z - R_p \sinh h}{Z\,\cancel{(\cosh h + \sinh h)}\;e^{h}}\right]$$

$$= \left[e^{h} - \frac{1}{2}\left(2 - \frac{R_p}{Z}\right)\sinh h\right]$$

$$= \left[e^{h} \cancel{\sinh h} - \cancel{1} + \frac{R_p}{2Z}\sinh h\right]$$

$$= \left[\cosh h + \frac{R_p}{2Z}\sinh h\right]$$

$$\therefore \cosh g = \cosh h + \frac{R_p}{2Z}\sinh h .$$

Z is almost real;

$$Z = \sqrt{\frac{L}{C}}\left(1 + \frac{R}{2i\omega L} - \frac{G}{2i\omega C}\right) \quad \text{for high } \omega.$$

h almost imaginary:

$$h = i\omega\sqrt{LC}\;s\left(1 + \frac{R}{2i\omega L} + \frac{G}{2i\omega C}\right).$$

$R_p = R_p + i\omega L_p \qquad L \gg R \quad \therefore R_p$ almost imag.

$\therefore$ if $g = ia + b$

$\cosh g = \cos a \cosh b + i \sin a \sinh b$

effect b small : $\cosh b \sim 1$, $\sinh b \sim b$

if $h = i\alpha + \beta \qquad \frac{R_p}{2Z} = \underline{X} + i\underline{Y}$

$\cosh h = \cos\alpha \cosh\beta + i \sin\alpha \sinh\beta$

$\sinh h = i \sin\alpha \cosh\beta + \cos\alpha \sinh\beta$

$\therefore \cos a + i\, b \sin a = (i \sin\alpha \cosh\beta + \cos\alpha \sinh\beta)(\underline{X} + i\underline{Y})$

$\therefore \cos a = \cos\alpha - Y \sin\alpha$ neglect $\bar{X}\beta$

& $b \sin a = \beta \sin\alpha + \beta Y \cos\alpha + \bar{X} \sin\alpha$

$$\therefore \cos a = \cos(\omega\sqrt{LC}\, s) - \frac{\omega L_p}{2\sqrt{L/C}} \sin(\omega\sqrt{LC}\, s) .$$

$a > \alpha$, since $\cos a < \cos\alpha$. i.e. phase shift

$$2\sin^2 a/2 = 1 - \cos a = 2\sin^2 \alpha/2 + \frac{\omega L_p}{\sqrt{L/C}} \sin\alpha/2 \cos\alpha/2$$

$$= \sin\alpha/2 \left[2\sin\alpha/2 + \frac{\omega L_p}{\sqrt{L/C}} \cos\alpha/2 \right]$$

Now $L_p \gg L_s$

$\alpha = s \cdot \omega\sqrt{LC}$ and $\frac{\omega L_p}{\sqrt{L/C}} = \frac{\omega L_p}{L_s}\sqrt{LC} \cdot s = \alpha \frac{L_p}{L_s}$

$$\therefore \frac{\omega L_p}{\sqrt{L/C}} \gg \alpha .$$

Hence if a is to be $< \pi$, α must be very small (comp. w.

Hence can expand in α :

$$\therefore 2\sin^2 a/2 = \alpha/2 \left[\alpha + \frac{\omega L_p}{\sqrt{L/C}} \right] = \frac{\alpha^2}{2} \left[1 + \frac{L_p}{L_s} \right]$$

$$\therefore \sin^2 a/2 = \frac{\alpha^2}{4} \left[\frac{L_s + L_p}{L_s} \right]$$

$$= \frac{1}{4} \omega^2 s^2 LC \left[\frac{L_s + L_p}{L_s} \right]$$

$$\sin a/2 = \frac{\omega}{2} \sqrt{sC(L_s + L_p)} = \omega/\omega_0$$

$$\omega_0 = \frac{2}{\sqrt{sC(sL + L_p)}}$$

Nov 5, 1941 If higher powers of α are taken into account:

$$2\sin^2 a/2 = \frac{\alpha}{2}\left(1-\frac{1}{6}\frac{\alpha^2}{4}\right)\left(\alpha + \frac{\omega L_p}{\sqrt{L/C}}\left(1-\frac{\alpha^2}{8}\right)\right]$$

$\therefore$ multiplied by factor $\left(1-\frac{1}{6}\alpha^2/4\right)\left(1-\alpha^2/8\right) = \left(1-\frac{1}{6}\alpha^2\right)$

$\therefore$ 2nd approx: $\sin a/2 = \frac{\omega}{\omega_0}\left(1-\frac{1}{12}\omega^2 LC_s\right)$

$LC_s^2 \approx \frac{4}{\omega_0^2}\frac{L_s}{L_p}$ $\quad\therefore \sin a/2 = \frac{\omega}{\omega_0}\left(1-\frac{1}{3}\frac{L_s}{L_p}\cdot\frac{\omega^2}{\omega_0^2}\right)$

(again if $L_p \gg L_s$, correction negligible).

Actually $\frac{L_p}{L_s} \approx 30$; $\therefore$ correction $\sim .01$ for $\omega \approx \omega$

$a \gg \alpha$; for $\omega \ll \omega_0$, $\sin a/2 \sim a/2 = \omega/\omega_0$

$$a = 2\omega/\omega_0 = \sqrt{C_s(L_s+L_p)}\,\omega$$

whereas $\alpha = \sqrt{C_s L_s}\,\omega$

$$\therefore a/\alpha = \frac{2}{\omega_0}\sqrt{\frac{C_s(L_s+L_p)}{C_s L_s}} = \cancel{\frac{2}{\omega_0}\frac{\omega_0}{2}\sqrt{\frac{L_p}{L_s}}}$$

$$= \sqrt{\frac{L_s+L_p}{L_s}} \approx 5 \text{ or } 6.$$

Velocity $c' = \frac{\omega s}{a}$

Flat line: $c' = \frac{1}{\sqrt{CL}}$ $\sim 200{,}000$ km/sec

Pupin line: $c' = \frac{1}{\sqrt{C\left(L+\frac{L_p}{s}\right)}}$ $\sim 25{,}000$ km/sec

much reduced.

Larger ω $\quad a > \frac{2\omega}{\omega_0}$, $c'_{\text{hi freq.}} < c'_{\text{low freq.}}$

i.e. dispersion.

Bad: hi freq. arrive later than low freq! So insert a filter-like element which compensates - i.e. delay low freq.

$b \ll \beta$; method: $b\sin a = \beta\sin\alpha + \beta Y\cos\alpha + \bar{X}\sin\alpha$

$$\beta = \tfrac{1}{2}\left(R\sqrt{\tfrac{C}{L}} + G\sqrt{\tfrac{L}{C}}\right) \qquad Y = \tfrac{1}{2}\omega L_p\sqrt{\tfrac{C}{L}}$$

$$\bar{X} = \tfrac{1}{2}L_p\sqrt{\tfrac{C}{L}}\left(\frac{R_p}{L_p} - \tfrac{1}{2}\frac{R}{L} + \tfrac{1}{2}\frac{G}{C}\right)$$

$$b\sin a = \tfrac{1}{2}\left(R\sqrt{\tfrac{C}{L}} + G\sqrt{\tfrac{L}{C}}\right)\left[s\omega\sqrt{LC} + \tfrac{s}{2}\omega L_p\sqrt{\tfrac{C}{L}}\right] + \tfrac{1}{2}L_p\sqrt{\tfrac{C}{L}}\left(\frac{R_p}{L_p} - \tfrac{1}{2}\frac{R}{L} + \tfrac{1}{2}\frac{G}{C}\right)\cdot s\omega\sqrt{LC}$$

AB.

$$= \tfrac{1}{2}(RC + GL)s^2\omega + \tfrac{1}{2}(R_pC + GL_p)s\omega$$

$$= \tfrac{1}{2}s\omega\left[C(Rs+R_p) + G(Ls+L_p)\right].$$

$$\therefore b = \frac{(\uparrow)}{\sin a} = \frac{(\quad)}{2\sin a/2 \cos a/2} = \frac{\tfrac{1}{2}s\omega\left[CR + GL\right]}{2\cdot\frac{\omega}{\omega_0}\cdot\sqrt{1-\frac{\omega^2}{\omega_0^2}}}$$

$$= \frac{1}{4}\frac{s\omega_0}{\sqrt{1-\frac{\omega^2}{\omega_0^2}}}\left[\quad\right]$$

$$\omega_0 = \frac{2}{\sqrt{Cs(Ls+L_p)}}$$

$$\therefore b/s = \cancel{\tfrac{1}{2}\sqrt{\frac{1}{Cs(Ls+L_p)(1-\frac{\omega^2}{\omega_0^2})}}\left[\quad\right]}$$

$$= \cancel{\tfrac{1}{2}\frac{1}{\sqrt{Cs(Ls+L_p)(}}}$$

$$b/s = \frac{1}{4}\sqrt{\frac{\omega_0^2}{1-\frac{\omega^2}{\omega_0^2}}}\left[\quad\right]$$

$$= \frac{1}{2\sqrt{1-\frac{\omega^2}{\omega_0^2}}}\left[\frac{C(R_s+R_p)}{\sqrt{Cs(Ls+L_p)}} + \frac{G(Ls+L_p)}{\sqrt{Cs(Ls+L_p)}}\right]$$

$$= \frac{1}{2\sqrt{1-\frac{\omega^2}{\omega_0^2}}}\left[\left(R_s+\frac{R_p}{s}\right)\sqrt{\frac{C}{L+L_p/s}} + G\sqrt{\frac{L+L_p/s}{C}}\right]$$

Just as would be expected by simple theory —

Frequencies near $\omega = \omega_0$ damped considerably — up to $.8\omega_0$, ok.

Nov. 7, 1941
G.A.

In order to transmit up to 7200 sec^{-1}:

If $C = 3.8\cdot10^{-8}$ F/km (1.4 mm diameter cable),

If $s = 1.7$ km, then $L_p = .02$ Henry

[$L\cdot s = .0012$ "].

this gives damping $b/s = .0186$ /km

[flat line $\beta/s = .0460$ /km]

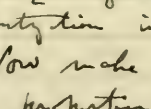
$c' \sim$ ~~20,000~~ 48,000 km/sec

Wave resistance is increased.

Can use a ferromagnetic substance; then inductance is increased by factor μ; damping reduced by factor $\sqrt{\mu}$.
e.g. Can wind iron wire around cable. Must have high μ, but small hysteresis loss: Permalloy works well

Principle of permalloy: It is known that hysteresis originates in the magnetic domains which are under strain, which gives preferred direction for magnetization; Hence to magnetize in a certain direction, must turn the domains, which requires much work; also when demagnetized, they don't return to original positions. So: small μ, large hysteresis. So to get good ferromagnetic, need one without strains; none such — so next best, one where strains don't influence magnetization.

Consider iron (Fe) bar ←▭→; if stretched in a certain direction, preferred magnetization direction is in direction of stretch; for Ni, preferred direction is opposite to direction of stretch [magnetostriction] [Converse also holds - i.e. magnetization in direction causes shrink || or ⊥ to direction] Now make an alloy of Fe & Ni (which is possible in any proportion); Plot magnetostriction vs. composition;



if crosses axis once - at that place, no magnetostriction, high μ, & low hysteresis.

Using this stuff, can get good transmission up to high frequencies & reduced damping.

So can use carrier frequency modulated by sound, instead of ~~transmitting~~ audio directly —

e.g. If frequencies up to 5000 are to be transmitted, can modulate a 50,000 sec^{-1}, say, wave by them; then have 45-55 kc. waves, found at end with a filter, or by beating with 50 kc oscillator; can do this also with carrier frequencies 40 & 60 kc; & so transmit many at once.

Low frequencies.

We had always before: $R \ll L\omega$; OK for $\omega > .5$ to 5 kc (flat)
$\omega > \sim 200$ sec^{-1} (perfect)

But not justified for 60 cycle transmission; even if there were increased L, which there isn't —

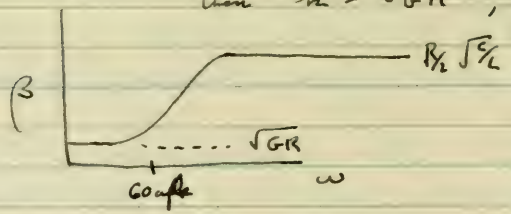
$$k = \sqrt{(R+iL\omega)(G+iC\omega)} \qquad G \ll C\omega \text{ even for } 60 \text{ sec}^{-1}$$

if $L\omega \ll R$, $k \approx \sqrt{i}\sqrt{CR\omega} = \sqrt{\frac{CR\omega}{2}}(1+i)$

$k_{real} \cong k_{im.}$; $\therefore$ as much damping as phase shift & decrease with decreasing frequency.

If ω so small that $G \gg C\omega$,

then $k = \sqrt{GR}$, real.



Nov. 10 '41 In actual line, amplifiers are put in after a decrease of about e^{-3} in intensity of signal

$$c' = \frac{\omega}{\alpha}; \quad \alpha = \text{I.P.}\sqrt{(R+iL\omega)(G+iC\omega)}$$

High freq: $\alpha = \omega\sqrt{LC}$, c' indep. of ω.

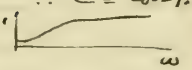
$$c' = \frac{1}{\sqrt{LC}}$$

Medium freq: $\alpha \sim \omega^{1/2}$ $\therefore c' \sim \omega^{1/2}$.

$\therefore c'$ decreases w. decreasing freq.

Low freq: $\omega < G/C$, $\alpha \sim \omega$ $\therefore c' =$ const. again.

$\therefore c'$ behaves about like β:



Behavior of current & potential at end of line.

Put impedance R_E at end of line.

At end: $u = I R_E$ $(x = x_0)$.

Along line: $I = a e^{kx} + b e^{-kx}$ $\qquad k = \sqrt{(R+iL\omega)(G+iC\omega)}$

$u = -Zu e^{kx} + Zb e^{-kx}$ $\qquad Z$-wave resistance $= \sqrt{\dfrac{R+iL\omega}{G+iC\omega}}$.

$$\therefore -Zu e^{kx_0} + Zb e^{-kx_0} = R_E(u e^{kx_0} + b e^{-kx_0})$$

$$a e^{kx_0}[-Z - R_E] = b e^{-kx_0}[R_E - Z$$

$$\therefore \frac{a e^{kx_0}}{b e^{-kx_0}} = \frac{Z - R_E}{Z + R_E}.$$

1) Line with open end: $R_E = \infty$,

$$\therefore a e^{kx_0} = -b e^{-kx_0} = -A$$

$$\therefore I = A e^{k(x_0 - x)} - A e^{k(x - x_0)}$$

$$= 2A \sinh k(x_0 - x).$$

Node at $x = x_0$, as it should.

Standing waves, with amplitude decreasing from beginning of line to end. if $k = i\alpha + \beta$,

then $\sinh k(x_0 - x) = \sinh\beta(x_0 - x)\cos\alpha(x_0 - x)$
$+ i\cosh\beta(x_0 - x)\sin\alpha(x_0 - x)$

For high freq: $\alpha \gg \beta$; several periods (back from end of line) before $\sinh\beta$ is as big as $\cosh$;

$$\therefore I = 2iA \sin\alpha(x_0 - x)$$

Going on toward beginning, amplitude increases almost exponentially; both terms equal

Maximum of $|I|$: $|I|^2 = \sinh^2\beta \cos^2\alpha + \cosh^2\beta \sin^2\alpha$

$= \sinh^2 - \sinh^2 \sin^2 + \cosh^2 \sin^2$

$= \sinh^2\beta + \sin^2\alpha$

I moving + reflected wave; near end of line, both ~ same amplitude, & give standing waves; but far away from end, one much larger than other; hence almost pure travelling wave (toward end). [at end, $|I|^2 \sim \sin^2$ - standing wave; at beginning, $|I|^2 \sim \sinh^2$; travelling wave with decreasing amplitude].

$I = \text{R.P.} : \left[2Ae^{i\omega t}\left(\sinh\beta(x_0-x)\cos\alpha(x_0-x) + i\cosh\beta(x_0-x)\sin\alpha(x_0-x)\right)\right]$

Near end: $I = -2A \sin\alpha(x_0-x) \sin\omega t$. standing wave

Far from end: $I = 2A \cdot \frac{1}{2} e^{\beta(x_0-x)} \cos[\omega t + \alpha(x_0-x)]$ travelling wave

Intermediate case for $\beta(x_0-x) \sim 1$.

Potential: $u = Z \cdot 2A \cosh k(x-x_0)$

$= 2AZ\left[\cosh\beta(x_0-x_0) \cos\alpha(x_0-x_0) + i\sinh\beta(x_0-x)\sin\alpha(x_0-x)\right]$

Near end of line: $u = 2AZ \cos\alpha(x_0-x)$

Far from end: $\bar{u} = AZ e^{\beta(x_0-x)} e^{i\alpha(x_0-x)}$.

$u = \text{R.P.}(\bar{u}e^{i\omega t}) = A|Z| e^{\beta(x_0-x)} \cos(\omega t + \alpha(x_0-x) + \varphi)$

where $Z = |Z| e^{i\varphi}$.

Near end: $u = 2A \cos\alpha(x_0-x) \cos(\omega t - \varphi)$

Phase lag $\pi/2 + \varphi$

both. I + u .

In this case, current node, potential loop at end.

Nov. 12, 1941

$$I = a e^{hx} + b e^{-hx}$$
$$U = -Z a e^{hx} + Z b e^{-hx}$$
$$\frac{a e^{hx_0}}{b e^{-hx_0}} = \frac{Z - R_e}{Z + R_e}$$

Case 2) : Short circuit : $R_e = 0$

then $a e^{hx_0} = b e^{-hx_0} = A$

So $I = 2A \cosh h(x_0 - x)$

$U = 2AZ \sinh h(x_0 - x)$

i.e. potential node, current loop at end.

Same as previous case; standing waves near end, travelling waves near beginning of line.

Case 3) : $Z = R_e$;

then $a = 0$;

$$I = A e^{h(x_0 - x)} = A e^{i\alpha(x_0 - x) + \beta(x_0 - x)}$$
$$U = A Z e^{h(x_0 - x)} = A Z$$
$$I e^{i\omega t} = A e^{i\alpha x_0} e^{\beta(x - x_0)} \underline{e^{i(\omega t - \alpha x)}}$$

Only travelling wave to end of line, & no reflected wave — this is important, because reflected wave is often undesirable, esp. in communication and in power transmission -

Krumhaus : Power line : Al. cable steel reinforced - 1" dia, 22 ft. apart; 220 kw @ 250 amp. at end; power factor .97 : 237 amp. at sending end.

If such $R_e = Z$, then takes 349 amp. at power station to give the same current at receiving end.

[$|Z+R_e| > |Z-R_e|$; because $Z = \sqrt{\frac{R+i\omega L}{G+i\omega C}}$.

complex phase of numerator can vary between 0 & $\pi/2$;

same for denom; ∴ ratio has phase $< |\pi/2|$;

So if $Z = |Z|e^{i\varphi}$, $-\pi/4 < \varphi < \pi/4$.

Now $R_e = R + iL\omega + \frac{1}{iC\omega}$; phase again $< |\pi/2|$

So if $R_e = |R_e|e^{i\psi}$, $-\pi/2 < \psi < \pi/2$;

$$\therefore |Z+R_e|^2 = |Z|^2 + |R_e|^2 + \cancel{2}\, Z R^* + Z^* R_e$$

$$|Z||R_e|e^{i(\varphi-\psi)} + |Z||R_e|e^{-i(\varphi-\psi)}$$

$$2|Z||R_e|\cos(\varphi-\psi)$$

if $\varphi = \pi/4$, $\psi = -\pi/2$ then $\varphi - \psi = 3\pi/4$

$\cos(\varphi-\psi) = -\frac{1}{\sqrt{2}}$;

then $|Z-R_e|^2 > |Z+R_e|^2$! and

reflected wave larger than incident

Power transmission:

$$P = IU = Z\left(b^2 e^{-2kx} - a^2 e^{2kx}\right)$$

$$\frac{IU(\text{receiving end})}{IU(\text{sending end})} \simeq e^{-2kx_0}\left(1 - \left(\frac{a e^{kx_0}}{b e^{-kx_0}}\right)^2\right)$$

Nov. 14'41 [Problem.

Perturbation calculation of damping.

$$(L_1\alpha^2 + R_1\alpha + \tfrac{1}{C_1})(L_2\alpha^2 + R_2\alpha + \tfrac{1}{C^2}) = M^2\alpha^4$$

Neglecting R_1, R_2, get $-\alpha^2 = \left\{\begin{matrix}\omega'^2 \\ \omega''^2\end{matrix}\right\} = \frac{\omega_1^2+\omega_2^2 \pm \sqrt{(\omega_1^2-\omega_2^2)^2 + 4k^2\omega_1^2\omega_2^2}}{2(1-k^2)}$

∴ i.e. $L_1L_2(1-k^2)(\alpha^2+\omega'^2)(\alpha^2+\omega''^2) = 0$.

Put in $R_1 R_2$; then $L_1L_2(1-k^2)(\alpha^2+\omega'^2)(\alpha^2+\omega''^2) = -R_1\alpha L_2(\alpha^2+\omega_2^2)$

$-R_2\alpha L_1(\alpha^2+\omega_1^2)$

let $\alpha = i\omega' + \beta$;

then $L_1L_2(1-k^2)\cdot\beta - 2i\omega'(\omega''^2-\omega'^2) = -2i\left[\omega'\beta_1 L_1L_2(\omega_2^2-\omega''^2) + \beta_2 L_1 (\omega_1^2-\omega''^2)\right]$

$\beta_1 = \frac{R_1}{2L_1}$ or $\beta = -\frac{\beta_1(\omega'^2-\omega_2^2) + \beta_2(\omega'^2-\omega_1^2)}{\ldots}$

for Tuned circuits: $\qquad -\alpha^2 = \dfrac{\omega^2(1 \pm k)}{1-k^2}$

$$\omega'^2 - \omega^2 = \frac{\omega^2 k}{1-k} \qquad \omega'^2 = \frac{\omega^2}{1-k}$$

$$\omega'^2 - \omega''^2 = \frac{2\omega^2 k}{1-k^2} \qquad \omega''^2 = \frac{\omega^2}{1+k} \;;$$

$$\therefore \beta' = -\frac{1}{2}\frac{\beta_1+\beta_2}{1-k} \leftarrow \text{large damping for high freq.}$$

$$\beta'' = -\frac{1}{2}\frac{\beta_1+\beta_2}{1+k} \leftarrow \text{small damping for low freq.}$$

Must see if β is negative:

$$\omega'^2 - \omega''^2 = \frac{\sqrt{}}{1-k^2}$$

$$\beta_1(\omega'^2-\omega_2^2) + \beta_2(\omega'^2-\omega_1^2) = (\beta_1+\beta_2)\left[\omega'^2 - \tfrac{1}{2}(\omega_1^2+\omega_2^2)\right] + (\beta_1-\beta_2)\left[\tfrac{1}{2}(\omega_1^2-\omega_2^2)\right]$$

it is !

Skin effect problem:

We had: $\dfrac{R_\omega}{R_0} = \dfrac{\lambda J_0(\lambda)}{2J_1(\lambda)}$, $\lambda = k r_0$

$= (1-i)\,k_0 r_0$.

To find setting in of skin effect.

r_0 = radius of wire

$k_0 = \sqrt{\dfrac{2\pi\sigma\mu\omega}{10^9}}$

Expand:

$$J_0 = 1 - \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4$$

$$J_1 = \frac{1}{2}\lambda\left(1 - \frac{1}{8}\lambda^2 + \frac{1}{192}\lambda^4\right).$$

$$\therefore \frac{\lambda/2\, J_0}{J_1} = \cancel{-\left(\frac{1}{4}-\frac{1}{8}\right)\lambda} = 1 - \frac{1}{8}\lambda^2 - \frac{1}{192}\lambda^4$$

$$\lambda^2 = -2i\,k_0^2 r_0^2$$

$$\lambda^4 = -4k_0^4 r_0^4$$

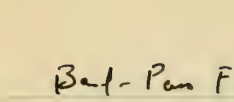
$$\therefore \frac{R_\omega}{R_0} = 1 + \frac{1}{4}i\,(k_0 r_0)^2 + \frac{1}{48}(k_0 r_0)^4$$

1st effect: increase of inductance upright; increasing of resistance.

Resistance increase becomes appreciable @ $\sim k_0 r_0 \sim 2.5$

Band-Pass Filter:

10/17/41



990 – 1010 kc.

$\omega_0 = 1{,}000$ kc

$$LC = L'C' = \frac{1}{4\pi^2 \cdot 10^{12}} = 2.5 \cdot 10^{-14}$$

Input cap. = 100 cm = $1.11 \cdot 10^{-10}$ F.

$$\frac{R_s}{2R_c} = -1$$

$$\therefore \; i\left(\omega L' - \frac{1}{\omega C'}\right) \cdot i\left(C\omega - \frac{1}{L\omega}\right) = -2$$

$$L'C\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 = 2$$

$$\therefore \; L'C = \frac{2}{\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}$$

Since $\omega \approx \omega_0$,

$$1 - \left(\frac{\omega_0}{\omega}\right)^2 \approx 2\left(1 - \frac{\omega_0}{\omega}\right) \qquad \therefore \; L'C \simeq \frac{1}{2\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}$$

$$\simeq \frac{1}{2(\omega^2 - \omega_0^2)^2}$$

$$LC = \frac{1}{\omega_0^2}$$

$$\therefore \; \frac{L'}{L} = \frac{1}{2}\left(\frac{\omega_0^2}{\omega^2 - \omega_0^2}\right)^2$$

$\omega_0 = 1000$, $\omega = 1{,}010$ $\quad \therefore \; \frac{L'}{L} = 5{,}000$.

$L' \gg L$, $C \gg C'$

$\therefore \; C' = 1.1 \cdot 10^{-10}$ F

$L' = 2.2 \cdot 10^{-4}$ H

$L = 4.4 \cdot 10^{-8}$ H

$C = 5.5 \cdot 10^{-7}$ F

At resonance,

$R_s = R'$

$$\frac{1}{R_c} = \frac{1}{Li\omega + R} + \frac{1}{i\omega C}$$

$$= -\frac{R}{(iL\omega)^2} = \frac{RC}{L}$$

$$\frac{R_s}{2R_c} = \sinh^2 \gamma/2 = \frac{R'RC}{2L} = \frac{1 \cdot .01 \cdot 5.5 \cdot 10^{-7}}{2 \cdot 4.4 \cdot 10^{-8}} = .62$$

$\sinh \gamma/2 = .25$, $\quad e^{\gamma} = 1.6 \quad e^{-\gamma} = .6$

For 250 kc, $\sinh^2 \gamma_2 \sim 25$, $\sinh \gamma_2 \sim 5$

Transform:

if $k = 1$,

get $\lambda_2 = \sqrt{\frac{L_1}{L_2}}$; $\lambda_k = \sqrt{\frac{L_2}{L_1}}$.

$\frac{U_2}{U_1} = \frac{I_1}{I_2} = \lambda$

$\therefore$ a factor ~ 100

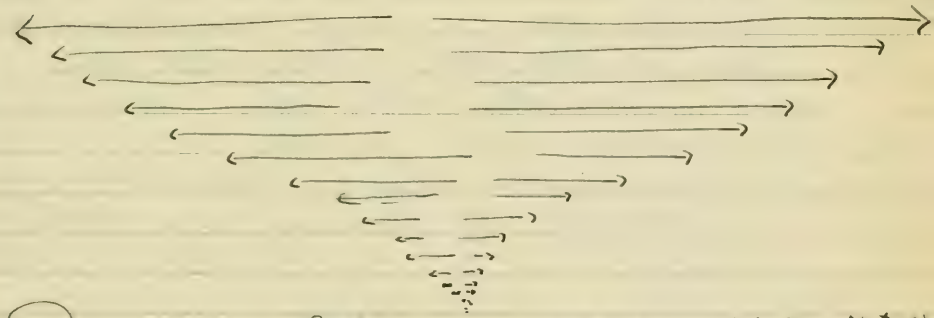
$(10^3)^n = (10^2)^n$

$\therefore n = 2.5$ factors req'd.

Light 250 kc reduced by 10^5.

$\frac{U_2}{U_1} = R_\infty$ $\quad$ U_1 determined by λ_∞

$\frac{U_2}{U_1} \simeq \lambda_\infty$ $\quad$ I_1 applicable for both a & b.



Power at end of line: $\quad I(t)U(t) = I' e^{i\omega t} U' e^{i(\omega t - \varphi)}$

Time average $= \frac{1}{2} I' U'^*$

$= \frac{1}{2} |I'|^2 R_E^*$

$= \frac{1}{2} |I'|^2 / R$ (Real part)

$U = R I$

$\left\{ = \frac{1}{2} |I'||U'| \frac{R}{|R|} \right.$

$\left. = \frac{1}{2} |I'||U'| \cos\varphi \right\}$.

Digression on complex quantities:

Power is really $\quad$ R.P. (I) · R.P. (U)

$= \text{RP}[I_1 + iI_i][\cos\omega t + i \sin\omega t] \cdot \text{R.P.}[U_R + iU_i][\cos\omega t - i \sin t]$

$= (I_1 \cos\omega t - I_i \sin\omega t)(U_R \cos\omega t - U_i \sin\omega t)$

and Power $= \frac{1}{2}(I_1 U_1 + I_i U_i)$

$= \frac{1}{2}$ R.P. $I_a U^*$

Power along line:

$$RI^2 + GU^2 .$$

$$= \left(-\frac{\partial U}{\partial x} - L\frac{\partial I}{\partial t}\right) I + \left(\frac{\partial I}{\partial x} - C\frac{\partial U}{\partial t}\right) U$$

$$= -\left\{ I\frac{\partial U}{\partial x} + U\frac{\partial I}{\partial x} + IL\frac{\partial I}{\partial t} + CU\frac{\partial U}{\partial t} \right\}$$

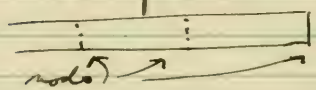
take average: $\emptyset$ terms in $\frac{\partial I}{\partial t}$, $\frac{\partial U}{\partial t}$ vanish,

$$\text{the Power} = -\overline{\left(I\frac{\partial U}{\partial x} + U\frac{\partial I}{\partial x}\right)}$$

$$= -\frac{\partial \overline{IU}}{\partial x} .$$

Nov. 19 '41 Standing waves: can get nodes + loops at any desired place by short circuiting at the right place; then U has node there, + other places: $U = A \sin \alpha(x_1 - x)$ if short circuit at x_1. $\alpha = \frac{\omega}{c'}$.

Can find nodes with neon tube; hence can measure wave length $\lambda = \frac{2\pi}{\alpha}$.



$$\lambda = \frac{2\pi}{\alpha} = \frac{2\pi c'}{\omega} = \frac{c'}{\nu} = \frac{c'}{c}\lambda_{vac} .$$

"Lecher wires"; used for obtaining dielectric constants at high freq; Put whole system in dielectric (e.g. a liquid), then $\alpha = \frac{\omega}{c'} = \omega\sqrt{LC}$

$$C_\varepsilon = C_1 \varepsilon \qquad C_\varepsilon = \text{cap. in diel.}$$
$$C_1 = \text{" " air}$$

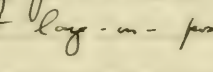
$$\therefore c'_\varepsilon = \frac{c'_1}{\sqrt{\varepsilon}} .$$

$$\therefore \lambda'_\varepsilon \sim \frac{1}{\sqrt{\varepsilon}} .$$

$\therefore$ Determine nodes in air, then in ε; ratio of λ's gives ratio of $\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$.

Large potential difference ~~attends~~ (e.g. cyclotron — want long-as-possible pot. diff. between dees).

To do this, try to make length of transmission line a quarter wave — (could use $3/4$ or any $\frac{2n+1}{4}$; but these are less stable —)



(voltage multiplication between excitation pt. & end.)

$$u = A \sin \alpha x \qquad \alpha l = \pi/2$$

$$l = \frac{\pi}{2\alpha} = \lambda/4$$

$$u_{end} = A; \qquad \lambda = \frac{2\pi}{\alpha} = 4l$$

$$u_{exc.} = A \sin \alpha x_0 = A \sin \frac{\pi x_0}{2l}$$

voltage multiplication by factor $\dfrac{1}{\sin \frac{\pi x_0}{2l}}$

The working impedance at the end, of course modified all then — (e.g. cyclotron dee capacity).

$$U = b Z e^{-ikx} - a Z e^{ikx}$$

$$\frac{a e^{ikl}}{b e^{-ikl}} = \frac{Z - R_E}{Z + R_E}$$

max. of u at end, only for $R_E \to \infty$.

$R_E \gg Z$ $\qquad R_E = \dfrac{1}{iC\omega} = \dfrac{1}{C\omega} e^{-i\pi/2}$ $\qquad Z = \sqrt{\dfrac{L'}{C'}}$ $\left.\begin{matrix} L' \\ C' \end{matrix}\right\}$ per unit length

define: $\tan \gamma = \dfrac{R_E}{iZ}$

$$\frac{a e^{ikl}}{b e^{-ikl}} = A e^{2i\gamma}, \quad a e^{ikl} = A e^{i\gamma}$$

$$b e^{-ikl} = A e^{-i\gamma}$$

then $U = 2 Z A i \sin[\alpha(l - x) + \gamma]$.

Resonance if $u = 0$ at $x = 0$

$\therefore \alpha l + \gamma = \pi$ $\qquad \therefore l = \dfrac{\pi - \gamma}{\alpha}$.

$$\frac{2\pi l}{\lambda'} = \pi - \gamma$$

$$l = \lambda'\left(\tfrac{1}{2} - \frac{\gamma}{2\pi}\right)$$

(if $R_c = \infty$, $\gamma = \pi/2$ then $l = \lambda'/4$ as before)

But since actually γ is between $\pi/2$ + π, length is shorter: $l < \frac{1}{4}\lambda'$.

Problem: Transmission line of concentric conductors:

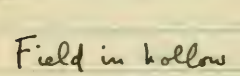
Assume $C_{diam} = 100$ cm.

take $1 > \frac{r_1}{r_2} > \frac{1}{e}$

30 meter waves.

Nov. 24 Field in hollow conductors

cylinder



Thick walls: thick compared with layer in which current flows.

Field E, H inside $\sim e^{i\omega t - i\gamma z}$

take coordinate z along cylinder. Interested in solution with $\gamma = \alpha - i\beta$, $\alpha \gg \beta$ (small damping). Phase velocity $\frac{\omega}{\alpha}$ · may be greater than c; (group velocity will never be).

In the vacuum: $\frac{1}{c}\dot{\underline{H}} = -\nabla \times \underline{E}$. $\nabla \cdot \underline{E} = 0$

$\frac{1}{c}\dot{\underline{E}} = \nabla \times \underline{H}$ $\nabla \cdot \underline{H} = 0$

In conductor: $\frac{1}{c}\dot{\underline{E}} + \frac{4\pi}{c}\underline{j} = \nabla \times \underline{H}$

neglect

At surface of conductor, tangential comp of E + H must be continuous

We expect tangential component of E to be very small, so as to avoid large losses. Hence for 1st approximation, say it vanishes. : $E_z = 0$, $E_\varphi = 0$ at surface of conductor.

No condition on E_r or $\underline{H}$.

Carson
Mead
Shelkunoff
Bell. Tel. Jour.
vol 15, p. 310 '36

$$(\nabla \times \underline{E})_z = \frac{1}{r}\frac{\partial}{\partial r}(r E_\varphi) - \frac{1}{r}\frac{\partial}{\partial \varphi} E_r$$

$$(\nabla \times E)_r = \frac{1}{r}\frac{\partial}{\partial \varphi} E_z - \frac{\partial E_\varphi}{\partial z}$$

$$(\nabla \times E)_\varphi = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}$$

Now $\frac{1}{c}\dot{H}_\varphi = -\frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r}$.

Put in time & z-dependence:

$$i\frac{\omega}{c} H_\varphi = i\gamma E_r + \frac{\partial E_z}{\partial r}$$

let $k = \frac{\omega}{c}$.

$$H_\varphi = \frac{\gamma}{k} E_r + \frac{i}{k}\frac{\partial E_z}{\partial r}$$

likewise $\frac{1}{c}\dot{E}_r = \frac{1}{r}\frac{\partial}{\partial \varphi} H_z - \frac{\partial H_\varphi}{\partial z}$

$$\text{or } -ik E_r = \frac{1}{r}\frac{\partial H_z}{\partial \varphi} + i\gamma H_\varphi$$

$$E_r = -\frac{i}{rk}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k} H_\varphi.$$

$$\therefore H_\varphi = \frac{\gamma}{k}\left[-\frac{i}{rk}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k} H_\varphi\right] - \frac{i}{k}\frac{\partial E_z}{\partial r}.$$

$$\therefore H_\varphi\left(1 - \frac{\gamma^2}{k^2}\right) = -\frac{i}{k}\left[\frac{\gamma}{k}\frac{1}{r}\frac{\partial H_z}{\partial \varphi} + \frac{\partial E_z}{\partial r}\right].$$

Let $\lambda^2 = k^2 - \gamma^2$

$$\lambda^2 H_\varphi = -i\left[k\frac{\partial E_z}{\partial r} + \frac{1}{r}\gamma\frac{\partial H_z}{\partial \varphi}\right].$$

Can also ~~eliminate~~ H_r: $E_r = -\frac{i}{rk}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k}\left[\frac{\gamma}{k} E_r - \frac{i}{k}\frac{\partial E_z}{\partial r}\right]$

$$\therefore E_r\left(1 - \frac{\gamma^2}{k^2}\right) = -\frac{i}{k}\left[\frac{1}{r}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k}\frac{\partial E_z}{\partial r}\right]$$

$$\therefore \lambda^2 E_r = -i\left[\frac{k}{r}\frac{\partial H_z}{\partial \varphi} + \gamma\frac{\partial E_z}{\partial r}\right]$$

Can get similar eq'ns with E, H interchanged:

$$\lambda^2 E_\varphi = -i\gamma \frac{\partial E_z}{\partial \varphi} + ik \frac{\partial H_z}{\partial r}.$$

and $$\lambda^2 H_r = -i\gamma \frac{\partial H_z}{\partial r} + \frac{ik}{r} \frac{\partial E_z}{\partial \varphi}.$$

So we have all r & φ components expressed in terms of z-comp.

To get an eq. for z-comp., use divergence condition:

$$\nabla \cdot \underline{E} = 0$$

i.e. $$\frac{\partial E_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{1}{r}\frac{\partial E_\varphi}{\partial \varphi} = 0$$

$$-i\gamma\lambda^2 E_z + \frac{1}{r}\frac{\partial}{\partial r}\left(-ik\frac{\partial H_z}{\partial \varphi} - i\gamma r \frac{\partial E_z}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left[-\frac{i\gamma}{r}\frac{\partial E_z}{\partial \varphi} + ik\frac{\partial H_z}{\partial r}\right] = 0$$

or: $$-i\gamma\lambda^2 E_z - ik\cdot\frac{1}{r}\frac{\partial^2 H_z}{\partial r\partial \varphi} - i\gamma\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - i\gamma\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \frac{ik}{r}\frac{\partial}{\partial \varphi}\frac{\partial H_z}{\partial r} = 0$$

cancel

$$\therefore\ i\gamma\lambda^2 E_z - i\gamma\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - i\gamma\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} = 0$$

or: $$\lambda^2 E_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} = 0$$

This is just wave eq'n, which would have come from $\nabla^2 \underline{E} = -k^2 \underline{E}$

Nov. 26, '41 $$\begin{cases} \nabla^2 E_z + k^2 E_z = 0 \\ \nabla^2 H_z + k^2 H_z = 0 \end{cases}$$

($\nabla^2 E_\varphi + k^2 \varphi = 0$ wrong!
$(\nabla^2 + k^2)$ ~ holds only for Cartesian coord's.

[because div grad $\underline{A} = \nabla^2 \underline{A}$ only in cartesian coords.

We had $\lambda^2 = k^2 - \gamma^2$: (γ gives z-dependence)

then $$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \lambda^2 E_z = 0$$

this gives $E_z = A\, J_m(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} m\varphi$

m an integer ≥ 0.

J_m must be an ordinary Bessel f'n, not a Hankel or Neumann function, since field must remain finite at $r=0$.

Also: $H_z = B\, J_m(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} m\varphi$.

Boundary conditions: $r = R$: (radius of tube)

$E_z = 0$, $E_\varphi = 0$ $\quad (E_\parallel = 0)$

We had: $$\lambda^2 E_\varphi = -i\gamma/r \frac{\partial E_z}{\partial \varphi} + ik\frac{\partial H_z}{\partial r}$$
$$= -\frac{i\gamma}{r} A\, J_m(\lambda r)\, m \begin{matrix}-\sin\\ \cos\end{matrix} m\varphi + ik B\, J_m'(\lambda r)\cdot\lambda \begin{matrix}\cos\\ \sin\end{matrix} m\varphi$$

To make $E_z = 0$, $r = R$: either $A = 0$

or $J_m(\lambda R) = 0$

in either case, 1st term of E_φ vanishes

To make 2nd " " " : either $B = 0$

or $J_m'(\lambda R) = 0$

Now $J_m(\lambda R) = 0$ and $J_m'(\lambda R) = 0$ cannot be simultaneously true (roots of Bessel f'n & derivatives do not coincide anywhere).

$\therefore$ if $J_m(\lambda R) = 0$, then $B = 0$
if $J_m'(\lambda R) = 0$, then $A = 0$

$\therefore$ 2 classes of solutions. 1) $H_z = 0$ identically "E sol'ns"
2) $E_z = 0$ " "H sol'ns"

{ For large x, $J_n(x) \sim \sin(x - (n - \frac{1}{2})\frac{\pi}{2})$

$J_n'(x) \sim \sin(x - (n - \frac{3}{2})\frac{\pi}{2})$

Zeros of J_n: $x = m\pi + (n - \frac{1}{2})\frac{\pi}{2}$

$= (m + \frac{n}{2})\pi - \frac{\pi}{4}$

all even n have same roots, odd n same roots.

Zeros of J_n' same as $J_{n \pm 1}$. }.

Since we are interested in modes with only a few nodes in radial direction, above doesn't apply. i.e. Important solutions are for small n & x.

Roots : E solutions

$m = 0$ (E, H, indep. of φ).

$J_0(\lambda R) = 0$ at $\lambda R =$ 2.4048 (Jahnke - Emde)

5.5201

⋮

$\boxed{\text{Call } (R\lambda_{m\mu}) \text{ the } \mu^{th} \text{ root of the } m^{th} \text{ Bessel f'n. i.e. } J_m(\lambda_{m\mu} R) = 0}$

$m = 1$

$J_1(\lambda R) = 0$: 3.8317

7.0156

H solutions

n ~~m=0~~ $= 0$

$J_0'(\lambda R) = 0$ $(J_0' = -J_1)$

$\therefore$ 3.8317 . (so in this case can have E + H sol'ns, at once)

7.0152

$n = 1$: $J_1'(\lambda R) = 0$ at $\lambda R =$ 1.84

$\pm$5.33

Each solution gives a characteristic field distribution which is easily calculated (pictures in Southworth, Bell Journal)

$$\lambda^2 E_r = -i\gamma \frac{\partial E_z}{\partial r} - \frac{i\omega}{r}\frac{\partial H_z}{\partial \varphi}$$

E_r is out of phase with E_z, or H_z

similarly H_r, H_φ, E_φ ; but are in phase with each other.

Possible values of k : $k^2 = \gamma^2 + \lambda^2$; γ almost real.

discrete set of possible λ's; $\lambda = \frac{x_{mn}}{R}$.

smallest $\lambda = \frac{1.84}{R}$ (H_{11} mode). $\boxed{\mathrm{Im}(x_{mn}) = 0}$

$\therefore k > \lambda_{mn} = \frac{1.84}{R}$ at least.

$\therefore$ only for suff. great frequencies can there be excited.

$l = \frac{2\pi}{\lambda_{osc}}$ $\lambda_{osc} = \frac{2\pi}{k} < \frac{2\pi}{\lambda_{mn}} = \frac{2\pi R}{x_{mn}}$.

biggest possible wave length $\sim \frac{2\pi}{1.84} R \sim 3.3 R$

no greater wave length can exist in these tubes.

Nov. 28 '41 For mode E_{01} :

$E_z = J_0(\lambda r)$ $H_z = 0$

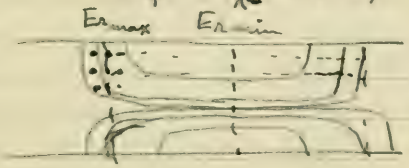
$E_r = \frac{i\gamma}{\lambda} J_1(\lambda r)$ $H_r = 0$

$E_\varphi = 0$ $H_\varphi = \frac{i k}{\lambda} J_1(\lambda r)$.

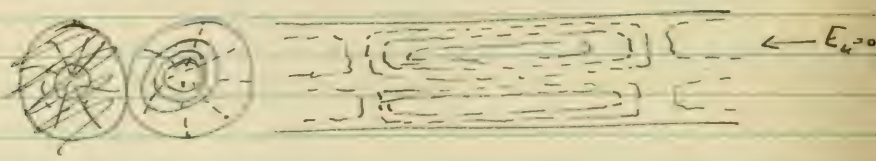
factor $e^{-i\gamma z}$ throughout

— E

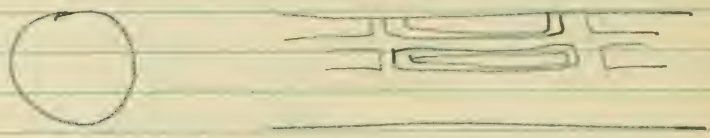
- - - H



H_{01} : same as previous except E & H interchanged — except that condition is now $E_\varphi = \frac{i\,k}{\lambda} J_1(\lambda l) = 0$ for $r = R$ (at wall) $\therefore$ $H_r = 0$ at wall, H_z a max. at wall;



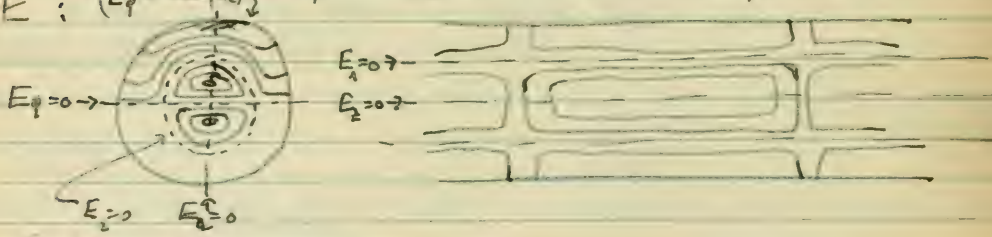
E_{02} :



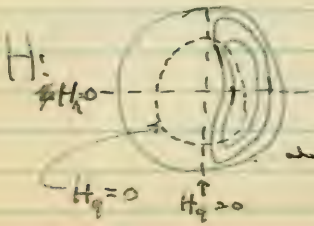
E_{11} :

$$E_z = J_1(\lambda r)\cos\varphi \qquad H_z = 0$$
$$E_r = -\frac{i\,k}{\lambda} J_1'(\lambda r)\cos\varphi \qquad H_r = \frac{i\,k}{\lambda^2 r} J_1 \sin\varphi$$
$$E_\varphi = -\frac{i\,k}{\lambda^2 r} J_1(\lambda r)\sin\varphi \qquad H_\varphi = \frac{-i\,k}{\lambda} J_1' \cos\varphi$$

E : ($E_\varphi = 0$ max. here)

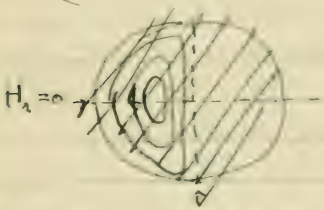


H:



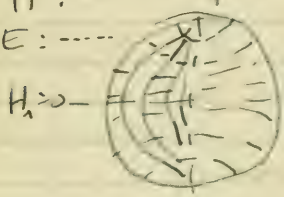
H_{11} (lowest possible λ : $\lambda R = 1.8$).

($J_1' = 0$ at wall). H lines:

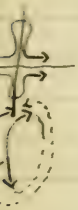


H : —— $H_\varphi = 0$

E : - - - -



(Southworth).



E_{01} wave looks something like putting a current going down axis — similar on shield.

E_{11} is like two currents going down pipe —

Dec. 1'41 E - wave:

$$\left.\begin{aligned} \lambda^2 E_r &= -i\gamma \frac{\partial E_z}{\partial r} \\ \lambda^2 E_\varphi &= -\frac{i\gamma}{r}\frac{\partial E_z}{\partial \varphi} \\ \lambda^2 H_r &= -\frac{ik}{r}\frac{\partial E_z}{\partial \varphi} \\ \lambda^2 H_\varphi &= ik\frac{\partial E_z}{\partial r} \end{aligned}\right\}$$

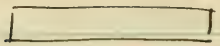
$$\begin{cases} \lambda^2 = k^2 - \gamma^2 \\ J_n(\lambda R) = 0 \end{cases}$$

$$k = \frac{\omega}{c} = \sqrt{\lambda^2 + \gamma^2}$$

For infinitely long line, any value of γ is allowed; otherwise depend on way the line is closed.

Suppose closed at both ends:

$z=0$ $\qquad\qquad$ $z=L$



at ends $E_r = E_\varphi = 0$.

$E_r, E_\varphi \sim \sin \gamma z$

$\therefore \gamma L = \cancel{\gamma}\, k\pi$ $\qquad$ k integer

$\gamma = \frac{k\pi}{L}$. $\qquad$ discrete set of γ's

but if $L \to \infty$, become practically continuous.

I infinite line:

may define phase velocity:

$e^{i(\omega t - \gamma z)}$

$$v_p = \frac{\omega}{\gamma} = \frac{ck}{\gamma} = c\,\frac{\sqrt{\gamma^2+\lambda^2}}{\gamma} > c.$$

$v_p \to \infty$ for $\gamma \to 0$

$v_p \to c$ for $\gamma \gg \lambda$.

But group velocity must not exceed c:

$v_g \neq v_p$ if v_p depends on ω.

$$v_g = \frac{d\omega}{d\gamma}. \qquad \omega = c\sqrt{k^2+\gamma^2}$$

$$= c\cdot\frac{\gamma}{\sqrt{\gamma^2+\lambda^2}} < c.$$

$v_p v_g = c^2$

consider a wave packet:

$\psi = \int d\gamma\, a(\gamma)\, e^{i(\omega t - \gamma z)}$

(only $\gamma \simeq \gamma_0$ occur) $\qquad$ choose $a(\gamma)$ so that ψ is large only for $z \simeq 0$ at $t=0$; ~~at a time~~

$12\Delta\gamma \gtrsim 1$

~~t th maximum~~

then $\psi \simeq e^{i(\omega_0 t - \gamma_0 z)}$

$$\int d\gamma\, a(\gamma)\, e^{i(\gamma-\gamma_0)\left(\frac{d\omega}{d\gamma}t - z\right) + \cdots}$$

$$= f\left[\frac{d\omega}{d\gamma}t - z\right]$$

Hence $v_g = \frac{d\omega}{d\gamma}$ in 1st approx.

If $E_z = J_m(\lambda r)\cos m\varphi$,

$E_\varphi = -i\,\frac{\gamma}{\lambda}\,\frac{J_m(\lambda r)}{\lambda r}\, m \sin m\varphi$

for low modes,

$$\frac{E_\varphi}{E_z} \sim \gamma/\lambda.$$

$$\frac{E_r}{E_z} \sim \gamma/\lambda$$

similarly H_φ, H_r;

$\therefore$ $\dfrac{\text{Transverse comp.}}{\text{longitudinal comp.}} \sim \gamma/\lambda$.

if $\lambda \sim \gamma$, all comp. are of same order of mag.;

But if $\gamma \gg \lambda$ Transverse comp. $\gg$ long. comp.

then get very nearly plane wave in z-direction with velocity very nearly vel. of light. — this case represents transition from line-waves to free space-waves.

E energy propagation along line.

Can be done two ways — (1) calculate energy content per unit length, + multiply by group velocity, gives total energy transmitted thru cross-section of tube per sec.

(2) calculate Poynting vector.

(1) Consider E-wave;

want $\frac{1}{8\pi}(E^2+H^2)$ per unit length:

$$\overline{W} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^R (E^2+H^2)\, r\, dr\, d\varphi$$

(don't forget we have been using complex representation)

$E = E' e^{i\omega t}$

$H = H' e^{i\omega t}$

$$\overline{W} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^R \tfrac{1}{2}\left((E')^2+(H')^2\right) r\, dr\, d\varphi$$

let $x = \lambda r$

$$= \frac{1}{16\pi}\int_0^{2\pi}\int_0^R \Big[\Big(J_n^2\cos^2 n\varphi + \frac{b^2}{\lambda^2}\cdot\frac{n^2}{x^2}J_n^2\sin^2 n\varphi + \frac{b^2}{\lambda^2}J_n'^2(x)\cos^2 n\varphi\Big) + \Big(\frac{b^2}{\lambda^2}\cdot\frac{n^2}{x^2}J_n^2\sin^2 n\varphi + \frac{b^2}{\lambda^2}J_n'^2(x)\cos^2 n\varphi\Big)\Big]\frac{x\,dx\,d\varphi}{\lambda^2}$$

$$= \frac{1}{16\pi}\cdot\frac{\pi}{\lambda^2}\int_0^R x\,dx\left[J_n^2 + \frac{b^2+b^2}{\lambda^2}\left(J_n'^2 + \frac{J_n^2 n^2}{x^2}\right)\right].$$

(2) Poynting vector: $S_z = \frac{c}{4\pi}(\underline{E}\times\underline{H})_z$ (per sq. cm)

$$= \frac{c}{4\pi}\left[E_r H_\varphi - E_\varphi H_r\right]$$

total transfer $= \frac{c}{4\pi}\cdot\frac{1}{2}\int_0^{2\pi}\int_0^R \left(E_r H_\varphi^* - E_\varphi H_r^*\right) r\, dr\, d\varphi$.

$$\therefore \text{total transfer} = \frac{c}{8\pi}\int\left(\frac{\gamma h}{\lambda^2}\left(\frac{\partial E_z}{\partial x}\right)^2 + \frac{\gamma h}{\lambda^2}\frac{1}{x^2}\left(\frac{\partial E_z}{\partial \varphi}\right)^2\right)\frac{x\,dx\,d\varphi}{\lambda^2}$$

$$= \frac{c}{8\pi}\cdot\frac{\gamma h}{\lambda^2}\int\left(J_n'^2\cos^2 n\varphi + \frac{1}{x^2}J_n^2 n^2\sin^2 n\varphi\right)\frac{x\,dx\,d\varphi}{\lambda^2}$$

$$= \frac{c}{8\pi}\cdot\pi\cdot\frac{\gamma h}{\lambda^4}\int\left(J_n'^2 + \frac{n^2}{x^2}J_n^2\right)x\,dx \quad .$$

$$\int J_n'^2\,x\,dx = \frac{J_n}{}\cdot x J_n'\Big|_0^{\lambda R} - \int_0^{\lambda R} J_n\cdot\frac{d}{dx}(x J_n')\,x\,dx$$

at λR, $J_n = 0$ for E-wave, $J_n' = 0$ for H-wave.

$$\therefore \int J_n'^2\,x\,dx = \int_0^{\lambda R} J_n\,[$$

$$\left\langle + \frac{n^2}{x^2}\right.$$

anyhow, $\int\left(J_n'^2 + \frac{n^2}{x^2}J_n^2\right)x\,dx = \int J_n^2\,x\,dx$

so by (1), get: $\overline{W} = \frac{1}{16\lambda^2}\int x\,dx\left\{1 + \frac{\gamma^2 + l^2}{\lambda^2}\right\}$

$$= \frac{1}{16\lambda^2}\cdot\frac{2l^2}{\lambda^2}\int J_n^2\,x\,dx$$

$$\overline{W}\cdot v_g$$

$$= \frac{l^2}{\cancel{8}\lambda^4}\cdot c\,\frac{\gamma}{h}\cdot\int J_n^2\,x\,dx$$

$$= \frac{\gamma h}{8\lambda^4}\int J_n^2\,x\,dx$$

+ this is same as result by (2) .

Dec 3, '41 Bethe forgot to set alarm.

Dec. 5, '41 We found Energy transport $S = Wc_g = W\frac{c\gamma}{k}$ W = energy/cm in tube.

$$S = \frac{c\gamma k}{8\lambda^4}\int_0^{x_0} J_m^2\, x\, dx. \qquad x_0 = \lambda R.$$

$J_m(x_0) = 0$ for E-wave

$J_m'(x_0) = 0$ for H-wave.

Now $\int_0^{x_0} J_m^2 x\, dx = \frac{1}{2}x^2\left(J_m^2 - J_{m-1}J_{m+1}\right)\Big|_0^{x_0}$

but $J_{m-1} = J_m' + \frac{m}{x}J_m$

$J_{m+1} = -J_m' + \frac{m}{x}J_m$

$$\therefore \int_0^{x_0} J_m^2\, x\, dx = \frac{1}{2}(x^2-m^2)J_m^2 + \frac{1}{2}x^2 J_m'^2\Big|_0^{x_0}$$

$$= \frac{1}{2}(x_0^2-m^2)J_m^2(x_0) + \frac{1}{2}x_0^2 J_m'^2(x_0)$$

for E-wave: $\frac{1}{2}x_0^2 J_m'^2(x_0)$

H-wave: $\frac{1}{2}(x_0^2-m^2)J_m^2(x_0)$

Energy consumed in wall per sec per cm of length: P

P/W = fraction of energy consumed per cm.

So that $E_z = E_{z_0}e^{-\beta z}$ $\beta = \frac{P}{2W}$.

(this valid because β will turn out to be $\ll \lambda$, k, etc.

Energy consumption due to currents in surface of tube — We had: $E_{tang} = 0$ at wall; but in general $H_{tang} \neq 0$; But H must vanish soon (beyond R) hence H decreases (exponentially) as we go into metal; there is a radial current $j = \sigma E$ on account of radial E; this causes a curl H, so that $H(r)$. Current exists in a very small sheath as comp. with size of tube; hence don't have to use Bessel f'ns — tube wall may be considered plane.

If η is cond. into metal: then is a current $j(x)$; and a Joule heat $\frac{1}{\sigma}\int j^2(x)\,dx\,dS$ (dS - surface elmt of tube), also have a mag. field $H_{tan}(x)$. Call y ~~the direction~~ of H.

$$j = \nabla \times H = \frac{\partial H_y}{\partial x} \qquad \therefore j \text{ in } z\text{-direction}$$

In metal: $\nabla^2 j = \frac{4\pi i \sigma \mu \omega}{c^2} j$ (not previous z - ⊥ to present x, y)

essentially:

$$\left(\frac{\sigma_{air}}{c^2} = \sigma_{air} = \sigma_{metal} \cdot 10^{-9}\right) \qquad \frac{\partial^2 j}{\partial x} = (\ )\, j$$

$$\therefore \quad j = B e^{\pm\sqrt{4\pi i \sigma \mu \omega}\, x}$$

let $\frac{1}{D} = \sqrt{2\pi\sigma\mu\omega}$

$$= B e^{\pm\sqrt{2\pi\sigma\mu\omega}\,(1+i)\,x}$$ throw out + sign

[$D \sim$ thickness current bearing sheath]

so that $\int j^2 dx = B^2 \int e^{-\frac{2x}{D}}$

$$= B^2 \int e^{-\frac{2x}{D}} = \frac{1}{2} B^2 D.$$

still need $B := j(0)$.

$$\nabla \times H = \frac{4\pi}{c} j = \frac{\partial H_y}{\partial x}$$

$$\frac{4\pi}{c}\int j\,dx = (H_y)_{x=0} - \underbrace{(H_y)_{x=\infty}}_{=0}$$

$$\therefore (H_y)_{x=0} = \frac{4\pi}{c} B D.$$

$$\therefore \text{Joule heat} = \frac{1}{\sigma} dS \cdot \frac{1}{2} D \cdot \left(\frac{c}{4\pi}\right)^2 \cdot \frac{1}{D^2} (H_{tan})^2$$

$$\text{total} \quad = \frac{c^2}{32\pi^2 \sigma D} \int H_{tan}^2\, dS \quad \text{over surface of tube.}$$

$$= \frac{1}{32\pi^2 \sigma D} \int H^2 R\, d\varphi\, dz$$

Power consumption per cm:

$$P = \frac{R}{32\pi^2 \sigma D} \int H_{tan}^2\, d\varphi$$

since $D \sim \frac{1}{\sqrt{\sigma}}$, $P \sim \frac{1}{\sqrt{\sigma}}$ $\therefore$ small loss for good conductor.

Now. $H_{tan}^2 = H_z^2 + H_\varphi^2$ E wave: $H_z = 0$, $H_\varphi \neq 0$

H-wave: ~~both~~ $H_z \neq 0$,

$$H_\varphi \sim \frac{\partial H_z}{\partial \varphi} \begin{cases} = 0, & n = 0 \\ \neq 0, & n \neq 0 \end{cases}.$$

Now $\frac{\text{trans}}{\text{long}} \sim \frac{h}{\lambda}$; ∴ trans. comp. increase with freq. relative to longitudinal; Hence at high freq., H_z is small, H_φ large; Hence the H_0 wave will be the best at high freq; smallest attenuation, since only $H_z \neq 0$ ~~at~~ for $n = 0$.

[But unstable for such high freq anyhow].

usually $h \approx \lambda$ – there it doesn't matter.

Dec. 8, '41 We had: Power consumption per cm.:

$$P = \frac{R}{32\pi^2\sigma D} \int H_{tan}^2 \, d\varphi$$

E waves: $H_{tan} = H_\varphi = \frac{il}{\lambda^2} \frac{\partial E_z}{\partial r}$.

$$E_z = J_n(\lambda r) \cos n\varphi$$

$$\frac{\partial E_z}{\partial r} = \lambda J_n'(\lambda r) \cos n\varphi$$

$$\therefore \int |H_{tan}|^2 d\varphi = + \frac{l^2}{\lambda^2} J_n'^2(x_0) \cdot \pi$$

$$\therefore P = \frac{1}{32\pi} \frac{R}{\sigma D} \frac{l^2}{\lambda^2} J_n'^2(x_0) \qquad x_0 = \lambda r$$

H waves: $H_{tan}^2 = H_\varphi^2 + H_z^2$

$$H_z = J_n \cos n\varphi$$

$$H_\varphi = -\frac{i\gamma}{\lambda^2} \frac{1}{r} \frac{\partial H_z}{\partial \varphi} = +\frac{i\gamma}{\lambda^2 r} J_n \, n \sin n\varphi$$

$$H_{tan}^2 = J_n^2 \left[\cos^2 n\varphi + \frac{\gamma^2 n^2}{\lambda^4 r^2} \sin^2 n\varphi \right]$$

$$\therefore \int H_{tan}^2 \, d\varphi = J_n^2 \left[1 + \frac{\gamma^2 n^2}{\lambda^4 r^2} \right] \pi$$

$$\therefore P = \frac{1}{32\pi}\,\frac{R}{\sigma D}\,J_n^2(x_0)\left(1 + \frac{l^2}{\lambda^2}\,\frac{n^2}{x_0^2}\right).$$

$P \sim \omega^2$ if λ is fixed + ω varied for E-wave;
For H waves: for $n = 0$; second term missing; smaller power consumption than for any other wave.

We had before: Energy transmitted per sec:

$$S = \frac{c\gamma l}{8\lambda^4}\cdot\frac{1}{2}\left[J_n^2(x_0^2 - n^2) + x_0^2 J_n'^2\right].$$

E waves: $J_n(x_0) = 0$

$$P/S = \frac{1}{2\pi}\cdot\frac{R}{\sigma D}\cdot\frac{l}{c\gamma}\cdot\frac{\lambda^2}{x_0^2} = \lambda^2/R^2$$

$$= \frac{l}{\gamma}\,\frac{1}{c\,2\pi\sigma D R}$$

but $D = \dfrac{1}{\sqrt{2\pi\sigma\mu\omega}}$

$$\therefore \sigma = \frac{1}{2\pi\mu\omega D^2}$$

$$\therefore P/S = \frac{l}{\gamma c}\,\frac{\mu\omega D}{R}.$$

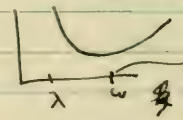
but $\omega/c = k^2$: $P/S = \dfrac{\mu k^2}{\gamma}\,\dfrac{D}{R}$

Don't use iron; makes large P/S

~~absorption~~ $\beta = \frac{P}{2S} \sim \omega^{1/2}$ at high freq.

(actually $\beta \sim \dfrac{\omega^{3/2}}{\sqrt{\omega^2 - \omega_0^2}} = f$ $\omega_0 = c\lambda$ for all ω

$$\frac{\partial f}{\partial \omega} = \frac{\frac{3}{2}\omega^{1/2}}{\sqrt{\ }} - \frac{\omega^{3/2}\cdot\omega}{(\sqrt{\ })^3} = 0$$



$1.732\,\omega_0$ (min.)

$$\frac{3}{2}(\omega^2 - \omega_0^2) = \omega^2$$

$$\therefore 3\omega^2 - 2\omega^2 = 3\omega_0^2$$

$$\omega = \sqrt{3}\,\omega_0$$

We found for $\bar{b}_0$ wave, smallest λ : $\lambda R = 2.4$

$\therefore k_{min} = \frac{2.4}{R} \cdot \sqrt{3}$ and wave lgth $\Lambda_{min} = \frac{2\pi}{k_{min}} = \frac{2\pi}{2.4\sqrt{3}} R$.

$\sim 1.45 R$.

also $\beta \sim \frac{D}{R}$ which is small. $\ll 1$.

we found before : $D = \frac{1}{\sqrt{2\pi\sigma\mu\omega}}$ take $\mu = 1$

$= \frac{1}{2\pi\sqrt{\sigma f}}$ $f = 2\pi\omega =$ freq.

for Cu : $D = \frac{1}{2\pi\sqrt{f}\sqrt{6\cdot 10^{-4}}}$

$= \frac{1}{.15\sqrt{f}}$

if $f \sim 10^{10}$, (3 cm waves)

$D = \frac{1}{.15\cdot 10^{5}} \sim 7\cdot 10^{-5}$ cm.

For best operation : $R \sim \frac{2}{3}\Lambda$

$\therefore R \sim 2$ cm.

and $D/R \sim 3.5\cdot 10^{-5}$ cm.

and $\beta \sim 2\cdot 10^{-5}$ cm.

Very small attenuation ; doesn't affect field very much at any one place. This justifies considering $\gamma + k^2$ real ; actually $\gamma = \alpha + i\beta$, $\lambda^2 = k^2 - \gamma^2 = k^2 - \alpha^2 - 2i\alpha\beta$, but these corrections are certainly negligible

$\beta \sim 2$ per km. about the same as what we found for ordinary lines at moderate freq. (~10

$\beta_{min} = \sqrt{\frac{2}{3}}\ \frac{2\pi}{\Lambda}\ \frac{1}{.15\sqrt{f}} \cdot \frac{1}{R}$ for Cu

$R \sim \Lambda \sim \frac{1}{f}$

$\therefore \beta_{min} \sim f^{3/2}$ for given mode.

This gets ~~worse~~ worse for high freq.

H-waves:

$\left[J_n'(x_0)=0\right]$

$$\frac{P}{S} = \frac{1}{2\pi c}\cdot\frac{R}{\sigma D}\cdot\frac{\lambda^4}{\gamma h}\cdot\frac{1+\frac{n^2\gamma^2}{x_0^2\lambda^2}}{x_0^2-n^2} \qquad x_0 = \lambda R$$

$$= \frac{1}{2\pi c}\cdot\frac{1}{\sigma D}\cdot\frac{\lambda^2}{\gamma h}\,\frac{x_0^2+\gamma^2 n^2}{(x_0^2-n^2)R}$$

$$= \frac{1}{2\pi c}\cdot\frac{1}{2\pi\sigma h D^2}\cdot D\,\frac{\lambda^2}{\gamma h}\,\frac{\lambda^2 x_0^2+\gamma^2 n^2}{(x_0^2-n^2)R}$$

$$= \frac{\sigma h}{c}\cdot\frac{D}{R}\cdot\frac{1}{\gamma h}\,\frac{\lambda^2 x_0^2+\gamma^2 n^2}{(x_0^2-n^2)}$$

for $n \neq 0$, $\gamma \gg \lambda$: $\quad \mu h\,\frac{D}{R}\cdot\frac{\gamma}{\lambda}\,\frac{n^2}{x_0^2-n^2}$.

$n=0$: $\quad \mu h\,\frac{D}{R}\cdot\frac{\lambda^2}{\gamma h}$.

for small γh, [illegible] E-wave (except $n=0$)

Dec. 10, '41 We found: E wave: $\beta \sim \frac{\omega^{3/2}}{\sqrt{\omega^2-\omega_0^2}}$

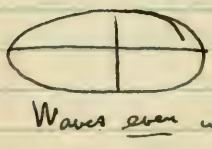
$n\neq 0$ H wave:

H_0 wave: $\beta \sim \frac{1}{\sqrt{\omega}\sqrt{\omega^2-\omega_0^2}}$

attenuation.

Elliptical cross-section: Chu, J. App. Phys. 9, 583.

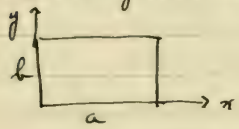
Two different types of waves: $\cos n\varphi$ & $\sin n\varphi$ no longer degenerate



λ is determined essentially by major axis in one case, by minor axis in the other.

Waves <u>even</u> with resp. to maj axis have higher λ than odd waves.

Rectangular cross-section. (Barrow - J. Opt. Phys.



Maxwell's eq'n's:

$$\frac{i\omega}{c} E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

$$-ik H_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}$$

$$(k^2 - \gamma^2) E_x = -i\gamma \frac{\partial E_z}{\partial x} - ik \frac{\partial H_z}{\partial y}$$

$$\lambda^2 H_y = -i\gamma \frac{\partial H_z}{\partial y} - ik \frac{\partial H_z}{\partial x}.$$

All quantities $\sim e^{i\omega t - i\gamma z}$

$(\nabla^2 + k^2) E_z = 0$

$E_z \sim e^{i\gamma z}$

$(\partial_{xy}^2 + \lambda^2) E_z = 0$

$\lambda^2 = k^2 - \gamma^2$.

$\lambda = \frac{\omega}{c}$.

Get for E_z:

$$E_z = A \sin \frac{\pi}{a} m x \sin \frac{\pi}{a} n x$$

E-wave

B.C.: $E_z = 0$ for: $\begin{cases} x = 0 & x = a \\ y = 0 & y = b. \end{cases}$

this gives: $E_x \sim \frac{\partial E_z}{\partial x} \sim \cos \ldots x \sin \ldots y$

$E_x = 0$ for $y = 0$, $y = b$

as it should.

$E_y \sim \frac{\partial E_z}{\partial y} \sim \sin \ldots x \cos \ldots y$

$E_y = 0$ for $x = 0$, $x = a$

as it must

Get also: $\lambda = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

solutions described by m + n: E_{mn}

[Cylindrical case: "quantum no's" were n (order of Bessel fn) and m (m^{th} root of Bessel fn)

H-wave: $H_z = A \cos \frac{\pi m x}{a} \cos \frac{\pi n y}{b}$.

then $E_x \sim \frac{\partial H_z}{\partial y}$ vanishes for $y = 0, b$ ✓

$H_x \sim \sin x \cos y$

$H_y \sim \cos x \sin y$.

$E_y \sim \frac{\partial H_z}{\partial x}$ " " $x = 0, a$ ✓

Here, cases $m = 0$, or $n = 0$ not admissible. $\lambda = \pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

smallest possible λ, if $a > b$, is $\lambda_{min} = \frac{\pi}{a}$ (H_{10} wave).

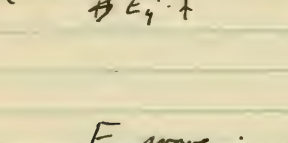
for E waves: $\lambda_{min} = \pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$ (E_{11} wave (or H_{11} wave))

H_{10} wave: $\quad H_z \sim \cos\frac{\pi x}{a}$

$E_x = 0 \quad E_z = 0$

$E_y \sim \sin\frac{\pi x}{a}$.

This is important wave; we can reduce b as much as we want, without changing a and hence λ_{min}:



Excite by electron beam:

transit time $= \frac{b}{v} \sim \frac{\pi}{\omega} = \frac{\pi}{c\lambda} = \frac{a}{c}$.

$\therefore$ need $a \gg b$.

$b \sim a\left(\frac{v}{c}\right) \sim \frac{1}{30}a$.

E wave:

again $\frac{b}{v} \sim \frac{b}{c}$, $\therefore$ can only excite H_{10} wave by electron.

(practically) impossible.

Power consumption: $\int H_{tan}^2\, dS = \int_0^a H_{tan}^2\, dx + \int_0^b H_{tan}^2\, dx$ (both surfaces)

Do this as a problem.

treat same way as cylindrical case; find transmission S.

and find attenuation $\beta = \frac{P}{2S}$.

H_{11} wave here is more nearly like H_1 for cylindrical case; J_0 (H_0-wave) corresponds more nearly to H_{11} wave here

Dec. 12'41 From wave guides we may go to:

(1) two concentric cylinders (conducts even at low frequencies; at high freq. behaves like the hollow tube we discuss (except need both J + N sol'n's). E_{tan} then must vanish at inner tube as well as outer. Treated well in ~~Stratton~~, E&M by Also a paper by Barrow ~~&~~ (Appl'ntal).

(2) Electromagnetic horns (wave guide of varying x-sect

e.g. a conical shaped conductor ; used as antenna ; has advantage over conventional antenna in smaller losses. We discuss later.

(3) Cavities.

Different from waveguide only in that z-dimension is of the same magnitude as other ; hence also B.C. on bounding planes in z-direction ; hence characteristic frequencies in 3-dimensions. Because they have resonant frequencies, can be used as radiators + detectors ; problems are (1) characteristic frequencies (2) (3) the damping Q.

Consider a parallelepiped (rectangular) ; sides a, b, c. in x, y, z direction.

B.C. :
$$\begin{cases} E_x = 0 \text{ at } z=0,\ z=c,\quad y=0,\ y=b. \\ E_y = 0 \text{ at } z=0,\ z=c,\quad x=0,\ x=a \\ E_z = 0 \text{ at } x=0,\ x=a,\quad y=0,\ y=b \end{cases}$$

Hence
$$\begin{cases} E_x = A_x \cos\frac{\pi l x}{a} \sin\frac{\pi m y}{b} \sin\frac{\pi n z}{c} \\ E_y = A_y \sin\frac{\pi l x}{a} \cos\frac{\pi m y}{b} \sin\frac{\pi n z}{c} \\ E_z = A_z \sin\frac{\pi l x}{a} \sin\frac{\pi m y}{b} \cos\frac{\pi n z}{c} \end{cases}$$

But we have also $\nabla \cdot \mathbf{E} = 0$;

whence $\frac{\pi l}{a} A_x + \frac{\pi m}{b} A_y + \frac{\pi n}{c} A_z = 0$

Let us call $\underline{k}$ the vector with components : $\left(\frac{\pi l}{a}, \frac{\pi m}{b}, \frac{\pi n}{c}\right)$

i.e. $\underline{k} \cdot \underline{A} = 0$. i.e. $\underline{A} \perp \underline{k}$.

we could write each component in terms of eight waves $e^{\pm \frac{i\pi l x}{a} \pm \frac{i\pi m y}{b} \pm \frac{i\pi n z}{c}}$. (transverse waves).

Notice also that $ck = \frac{\omega}{c}$. $\left[\left(\nabla^2 + \frac{\omega^2}{c^2}\right) E_i = 0 \text{ where } k^2 = \frac{\omega^2}{c^2}\right.$.

$\frac{\omega^2}{c^2} = k^2 = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$. Can't have all three of $l, m, n = 0$.

Nor even two ; not more than one $= 0$.

e.g. if $l=0$, $E_x \neq 0$, $E_y = 0 = E_z$, i.e. plane polarized standing wave.

Now we need the magnetic field:

$$-ik H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

$$= (k_x A_y - k_y A_x) \cos\frac{\pi l x}{a} \cos\frac{\pi m y}{b} \sin\frac{\pi n z}{c}$$

$$\therefore \underline{H} = \frac{i(\underline{k} \times \underline{A})}{k} \cos\cos\sin ; \quad \text{[sin goes with coordinate of H this}$$

$\underline{k} \times \underline{A}$ is $\perp$ to k & A; $\therefore$ E, H, k, are mutually $\perp$.

$\nabla \cdot H = 0$ automatically;

for $\nabla \cdot H = i[\underline{k} \cdot (\underline{k} \times \underline{A})] \frac{1}{k} \cos\cos\sin = 0$.

smallest value of k: if c is smallest dimn of box, then take $n=0$, $m=1$, $l=1$, and $k^2 = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$.

$$k = \frac{\pi}{ab}\sqrt{a^2+b^2}.$$

wavelength $\Lambda = \dfrac{2ab}{\sqrt{a^2+b^2}}$.

longest wavelength if $a=b$:

$$\Lambda_{max} = \sqrt{2}(a).$$

Dec. 15.

(Jean + Perkins) Let P = power consumption / sec.

W = total e.m. energy in cavity.

$\beta = \frac{P}{2W}$; i.e. fields decay as $e^{-\beta t}$.

$W = \frac{1}{8\pi}\int(E^2+H^2)\,d\tau$ For a plane wave $\bar{E} = \bar{H}$;

our sol'n is a superposition of plane waves, Hence $\int E^2 = \int H^2$,

and $W = \frac{1}{4\pi}\iiint \bar{H}^2\,d\tau$.

Power consumption is $P = \frac{1}{\sigma}\int \underline{j}^2\, d\tau$

$$= \frac{1}{\sigma}\int j^2\, d\sigma\, dn$$

$\int j^2 dn = \frac{1}{D}\left(\int j\, dn\right)^2$ $\quad D$ = thickness of current sheath

$D \approx \frac{1}{\sqrt{2\pi\sigma\mu\omega}}$

[Walls > .01 mm thick may be considered infinitely thick.]

Now $4\pi \int j\, dn = \underline{n} \times \underline{H}$

$j \sim e^{-\frac{n}{D}(1+i)}$ $\qquad \int j\, dn = \frac{D}{1+i}$

$|j|^2 \sim e^{-\frac{2n}{D}}$ $\longrightarrow$ $\left|\int j\, dn\right|^2 = \frac{D^2}{2}$.

$\int j^2 dn = \frac{D}{2}$.

$$\therefore P = \frac{1}{\sigma}\int j^2 d\tau$$

$$= \frac{1}{\sigma D}\int \overline{\left(\int j\, dn\right)^2} d\sigma$$

$$= \frac{1}{(4\pi)^2 \sigma D}\int \overline{(n \times H)^2}\, d\sigma$$

$$= \frac{1}{(4\pi)^2 \sigma D}\int \overline{H_{tan}^2}\, d\sigma$$

$$= \frac{2\pi\mu\omega D^2}{(4\pi)^2 D}\int \overline{H_{tan}^2}\, d\sigma$$

$$= \frac{\mu\omega D}{8\pi}\int \overline{H_{tan}^2}\, d\sigma$$

but $\because$ $h\, H_n = (\text{curl } E)_n = 0$

$$\therefore \bar{P} = \frac{\mu\omega D}{8\pi}\int \bar{H}^2\, d\sigma$$

$$\beta = \frac{P}{2W} = \frac{\mu\omega D}{4}\,\frac{\int H^2 d\sigma}{\int H^2 d\tau} \sim \frac{\omega D}{L}.$$

$$Q = \frac{\omega}{2\beta} = \frac{2}{\mu D}\frac{\int H^2 d\tau}{\int H^2 d\sigma} \sim \frac{L}{D}.$$

For a rectangular cavity: let $l, m, n \neq 0$.

then $\int |H|^2 d\tau = \frac{1}{8} B^2 \cdot V$. $\quad V = \text{volume}$. $\quad \underline{B} = \frac{\underline{k} \times \underline{A}}{|k|}$.

$$\int_{xy\text{ surface}} H^2 d\sigma = \frac{1}{2} B_x^2 a(b+c).$$

whole surface: $\frac{1}{2} B_x^2 a(b+c) + \frac{1}{2} B_y^2 b(a+c) + \frac{1}{2} B_z^2 c(a+b)$

$$\therefore \frac{1}{Q} = \frac{\mu D}{2} \cdot \frac{\frac{1}{2}}{\frac{1}{8}V} \left\{ \frac{B_x^2}{B^2} a(b+c) + \frac{B_y^2}{B^2} b(a+c) + \frac{B_z^2}{B^2} c(a+b) \right\}$$

$V = abc$

$$= \frac{2\mu D}{B^2} \left\{ B_x^2 \left(\frac{1}{b} + \frac{1}{c}\right) + \cdots \right\}.$$

for $B_x = B_y = B_z$, $\quad a = b = c$,

$$\frac{1}{Q} = \frac{2\mu D}{B^2} \left\{ B_x^2 \left(\frac{2}{a}\right) + B_y^2 \left(\frac{2}{a}\right) + \cdots \right\}$$

$$= \frac{4\mu D}{a}, \quad Q = \frac{a}{4\mu D}.$$

If $n = 0$, $l, m \neq 0$, $\quad \int |H|^2 d\tau = \frac{1}{4} B^2 V$

$$\int_{xy} |H|^2 d\sigma = (B_x^2 + B_y^2) \cdot \frac{1}{2} ab$$

$xy \qquad B_x^2\, ac$

$yz \qquad B_y^2\, bc$

$$\therefore \frac{1}{Q} = \frac{\mu D}{2} \cdot \frac{B_x^2\left(ac + \frac{ab}{2}\right) + B_y^2\left(bc + \frac{1}{2}ab\right)}{\frac{1}{4}B^2 \cdot abc}$$

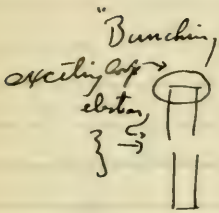
$$= \mu D \left\{ \frac{B_x^2}{B^2}\left(\frac{2}{b} + \frac{1}{c}\right) + \frac{B_y^2}{B^2}\left(\frac{1}{c} + \frac{2}{a}\right) \right\}$$

If $c \ll a, b$, $\frac{1}{Q} \sim \frac{D}{c}$ or $Q = \frac{c}{D}$. in general with large Q.

Problem: Study H_z; distrib. for lowest mode with square base

Dec. 17. $Q \sim \frac{\text{dimension}}{D}$. Transit time for electron: $\leq \frac{1}{2}$ Period

"Bunching" - Varian, Webster, J. App. Phys. '39.



electrons are alternately accelerated & decelerated in cavity; results in groups of electrons coming out.
(Field alternately with & against electron)

If these "bunched" electrons are sent thru another cavity, it will be excited; thus the oscillations are harmonically & can even be amplified quite considerably.

Transit time: if g is width of cavity: $= g/v$.

Period of oscillation $= \Lambda/c$.

$\therefore \frac{g}{v} \leq \frac{\Lambda}{2c}$

$\therefore g \leq \Lambda \frac{v}{2c} = \Lambda \sqrt{\frac{E}{2mc^2}}$

$2mc^2 = 1$ Mev.

if $E = 400$ v.,

$g \leq \Lambda \cdot \frac{20}{1000} = \frac{\Lambda}{50}$.

i.e. $g \leq$ other dimensions of cavity.

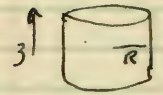
If we have, e.g., dimension a, a, g, with $g \leq a$, then the lowest mode is with $E \parallel g$; and $\Lambda = a\sqrt{2}$; (convenient).

$\therefore g \leq a \sqrt{\frac{E}{mc^2}}$.

i.e. $g \sim \frac{1}{30} - \frac{1}{10} a$.

But this reduces Q. [want large Q for: sharp resonance, high signal to noise ratio; & greater amplification. smaller power loss (if used as emitter).

Consider a cylindrical cavity:



get E waves : $E_z \neq 0$, $H_z = 0$

H waves : $H_z \neq 0$, $E_z = 0$.

For E waves: $E_z = A \cos\frac{\pi l}{a} z \; J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} n\varphi$

(with $J_n(\lambda R) = 0$). (n, l, integers).

Use had: $\lambda^2 E_r = -i\left(\frac{\partial E_z}{\partial r} - \frac{il}{r}\frac{\partial E_z}{\partial \varphi}\right)$.

$E_r = 0$ at $z=0$, $z=a$ $\therefore \frac{\partial E_z}{\partial r} = 0$ at $z=0, a$.

$\therefore \frac{\partial E_z}{\partial r} \sim \sin z$

$E_z = A \cos\frac{\pi l z}{a} J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} n\varphi$.

$E_r \sim \sin\frac{\pi l z}{a} J_n'(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} n\varphi$

$E_\varphi \sim \sin\frac{\pi l z}{a} J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} n\varphi$ $\quad n = 0, 1, 2 \ldots$

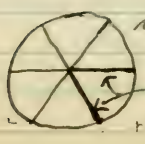
$H_r \sim \cos\frac{\pi l z}{a} J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} n\varphi$ $\quad m^{th}$ root of J

$H_\varphi \sim \cos\frac{\pi l z}{a} J_n'(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} n\varphi$ $\quad l = 0, 1 \ldots$

if $l=0$, E is in z-direction – other comp. vanish. This corresponds to mode used with rectangular box on previous page. Take 1st root of J_0 also, then E_z falls from max at middle to zero at sides.

Can make $a \ll R$, or let electron go thru in z-direction; not very different from previous problem –

But if $n \neq 0$, E_z has nodes where $\begin{matrix}\cos\\ \sin\end{matrix} = 0$; look at cross section of cylinder: let $n=3$. consider nodes of $\cos 3\varphi$.



Can put a metal wall in where nodes are; situation isn't changed, since E_z vanishes there anyhow. Only effect is to lower Q.

Hence can make this shape: and can send electrons thru narrow part (in plane of paper). So there is now a small dimension thru which to send electrons, and a large dimension to determine wavelength.

also $Q \sim \frac{\int |H|^2 d\tau}{\int |H|^2 d\sigma} \sim r$

In fact now n doesn't have to be integer;

Consider an H-wave: $H_z = \sin^n \frac{lz}{a} J_n(\lambda r) \cos n\varphi$

$E_z = 0$

$E_\varphi = \quad J_n'(\lambda r) \cos n\varphi$

$E_r = \quad J_n(\lambda r) \sin n\varphi$

Also, we can use other Bessel f'n's, because we have left out the center.

Dec. 19. Above is not so. — can't get large oscillations by electrons. ——

Spherical Cavity:

Much work done on spherical elmg fields in optics, e.g. scattering of light by conducting spheres, etc.

[Mie ~ 1908]
[Debye - 1909]
[Born - Optik p. 275]

Maxwell eq'ns, in polar coord's.

$$ik E_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(r \sin\theta H_\varphi) - \frac{\partial}{\partial\varphi}(r H_\theta)\right]$$

$$ik E_\theta = \frac{1}{r \sin\theta}\left[\frac{\partial H_r}{\partial\varphi} - \frac{\partial}{\partial r}(r \sin\theta H_\varphi)\right]$$

$$ik E_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial\theta}\right]$$

$-ikH$ = same with E instead of H.

Distinguish between: E waves: $E_r \neq 0$, $H_r = 0$

H " : $H_r \neq 0$, $E_r = 0$

Boundary cond: $E_\varphi = E_\theta = 0$ at $r = R$.

E waves: $H_r = 0$, $\therefore \frac{\partial}{\partial\theta}(r\sin\theta\, E_\varphi) = \frac{\partial}{\partial\varphi}(r E_\theta)$.

Then $-ik H_\varphi = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(r E_\theta)}$ $= \cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial\varphi}\right)}$.

i.e. $\begin{cases} r E_\theta = \frac{\partial u}{\partial\varphi} \\ r\sin\theta\, E_\varphi = \frac{\partial u}{\partial\varphi} \end{cases}$

$$ik E_\theta = \frac{-1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta\, H_\varphi)$$

E_θ eq'n:

$$\therefore ik\sin\theta\frac{\partial u}{\partial\theta} = -\frac{\partial}{\partial r}(r\sin\theta\, H_\varphi) =$$

Let $u = \frac{\partial}{\partial r}(r\Pi) \Rightarrow -ik\sin\theta\frac{\partial^2}{\partial\theta\partial r}(r\Pi)$

E_φ eq'n

$$\therefore H_\varphi = -\frac{ik}{r}\frac{\partial}{\partial\theta}(r\Pi) = -ik\frac{\partial\Pi}{\partial\theta}.$$

$$\therefore \frac{\partial}{\partial r}(r H_\theta) = \frac{ik}{\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}(r\Pi)$$

$$H_\theta = \frac{ik}{\sin\theta}\frac{\partial\Pi}{\partial\varphi}.$$

E_r eq'n. and $E_r = -\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\frac{1}{\sin\theta}\frac{\partial\Pi}{\partial\varphi}\right]$.

$= -\frac{1}{r}\Omega\Pi$ [Ω is angular part of ∇^2 in sph. coord

All quantities are now expressed in terms of Π.

Haven't used yet the H_θ & H_φ eq'n;

E_θ eq'n: $-ik H_\theta = \frac{1}{r\sin\theta}\left(\frac{\partial E_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\sin\theta\, E_\varphi)\right)$

$$\therefore +\frac{k^2}{\sin\theta}\frac{\partial\Pi}{\partial\varphi} = \frac{1}{r\sin\theta}\left(-\frac{1}{r}\frac{\partial}{\partial\varphi}\Omega\Pi - \frac{\partial^2 u}{\partial\varphi\partial r}\right)$$

$$= \frac{-1}{r\sin\theta}\left(-\frac{1}{r}\frac{\partial}{\partial\varphi}\Omega\Pi - \frac{\partial^3(r\Pi)}{\partial r\partial\varphi\partial r}\right)$$

$$\therefore \frac{\partial}{\partial\varphi}\left[k^2\Pi + \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Pi) + \Omega\Pi\right] = 0 \quad \frac{\partial}{\partial\varphi}\left[k^2 + \frac{\partial^2}{\partial r^2}\right]\Pi = 0$$

H_φ eq'n: $$-ikH_\varphi = \frac{1}{r}\left(\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right]$$

$$-k^2\frac{\partial \Pi}{\partial \theta} = \frac{1}{r}\left(\frac{\partial}{\partial r}\frac{\partial u}{\partial \theta} - \frac{\partial}{\partial \theta}\left(-\frac{1}{r}\Omega\Pi\right)\right)$$

$$\cancel{= \frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}(r\Pi)}$$

$$\therefore \frac{\partial}{\partial\theta}\left[k^2+\nabla^2\right]\Pi = 0 \quad \text{again.}$$

If Π depends only on r, all components of field vanish; hence $(k^2+\nabla^2)\Pi = 0$ and Π satisfies wave eq'n.

B.C. $$\frac{\partial}{\partial r}(r\Pi)_{r=R} = 0 .$$

H wave: interchange E + H, + change sign of k. Then B.C. is $\Pi = 0$ for $r = R$.

Solutions of wave eq'n:
$$\Pi = \frac{\psi(r)}{r} P_{lm}(\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\varphi .$$

Equation for ψ: $$\frac{d^2\psi}{dr^2} + \left(k^2 - \frac{l(l+1)}{r^2}\right)\psi = 0 .$$

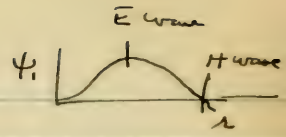
Solution: highbrow: $\psi \sim \sqrt{r}\, J_{l+\frac{1}{2}}(kr)$

lowbrow: $\sim \sin kr\, f_1(kr) + \cos kr\, f_2(kr)$

(f_1 + f_2 polynomials)

$l = 0$: no solution: because this gives $\Pi = \Pi(r)$ only, which gives no field. [i.e. no solutions to all'ing eq'ns with vanishing angular momentum — by angular momentum is meant $\int(\underline{r}\times\underline{p})\,dV$; momentum $\underline{p} \sim \underline{S}$ - (Poynting vector) turns out always $\sim l$.]

$l=1$: get $\psi = \frac{\sin kr}{kr} - \cos kr$.



E wave sol'n: $kR \sim 2.5$

H wave sol'n: $kR \sim 4.0$

$l=1,\ m=0$ $\quad \Pi = \frac{1}{r}\left(\frac{\sin kr}{kr} - \cos kr\right)\cos\theta$

then $E_\theta \sim \sin\theta \sim H_\varphi$

$E_\varphi = 0 = H_\theta$

$E_r \sim \cos\theta$

$l=1,\ m=\pm 1$: same: get $\sin\theta \begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}$ which means just replace z-direction by x or y direction.

Jan 5 1948 We had:

$$E_\theta = \frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r\,\partial\theta}$$

$$E_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2(r\Pi)}{\partial r\,\partial\varphi}$$

$$H_\varphi = \frac{ik}{r}\frac{\partial(r\Pi)}{\partial\theta}$$

$$H_\theta = \frac{-ik}{r\sin\theta}\frac{\partial(r\Pi)}{\partial\varphi}.$$

$$E_r = -\frac{1}{r}\Omega\Pi$$

$$H_r = 0$$

$$\nabla^2(\Pi r) = 0$$

$$\therefore \Pi = \frac{\psi(r)}{r} P_{lm}(\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi .$$

E waves:

$$\left(\frac{\partial\psi}{\partial r}\right)_R = 0 .$$

H waves:

interchange E & H in equations, & displace k by $-k$.

and $\psi = 0$ at $r = R$

In general $\psi \sim \sqrt{r}\, J_{l+\frac{1}{2}}(kr)$.

We found mode $l=0$ doesn't exist.

$l=1$: see top of page.

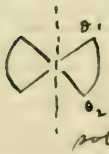
these solutions have same form as 2^l-pole radiation

i.e. $l=1$: E-wave $\rightarrow$ electric dipole

$l=1$: H-wave $\rightarrow$ magnetic dipole.

[in case of radiation, assume radiating charges etc. at origin, & use Hankel function instead of Bessel fn: $\psi \sim \sqrt{r}\, H^{(1)}_{l+1/2}(r)$:

Consider sphere with cone cut out.



Old solutions still hold; but also get new ones, because poles are not included; hence solutions don't have to behave at $\theta = 0$ & π; in this case the Legendre equation has solutions Q_l.

$$Q_0 \sim \frac{1}{2} \ln \frac{1+\cos\theta}{1-\cos\theta}$$

$$Q_l \sim P_l Q_0 + (\cos\theta)^{l-1} + \dots$$

Now the mode $l=0$ is no longer missing, since (Π/r) is no longer independent of θ for $l=0$.

$l=0$: $\quad r\Pi = -\sin kr\, Q_0(\theta)$

since $r\Pi$ is indep. of φ, we get $E_\varphi = 0 = H_\theta$

but $E_\theta = \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta}(r\Pi)$

$$= \frac{1}{r} \cdot k \cos kr \cdot \frac{1}{2}\left(\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta}\right)(-\sin\theta)$$

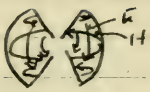
$$= +\frac{k}{r} \frac{\cos kr}{\sin\theta}.$$

$$H_\varphi = -\frac{ik}{r} \sin kr \cdot -\frac{1}{\sin\theta}$$

$$= \frac{ik \sin kr}{r \sin\theta}.$$

$$E_r = \frac{1}{r}\Omega(r\Pi) = \frac{1}{r}\Omega(\psi Q_l) = l(l+1)\frac{\psi}{r}.$$

$\therefore\ l=0:\ E_r = 0$



$E_\theta = 0$ at surface:

$\cos kR = 0, \quad kR = \pi/2 = 1.57$

$\therefore$ frequency is lower for this mode than for any of those of whole sphere.

On cones: $E \perp$ surface, hence B.C. satisfied.

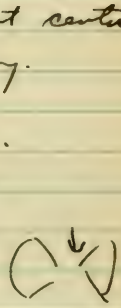
for $l \neq 0$, E_r in general $\neq 0$; must make $E_r(\theta_1) = 0$.
this will require a linear combination $E_r(\theta_2) = 0$.
$P_l + \alpha Q_l$ and in general
not any θ_1 & θ_2 will be possible. — will be related.

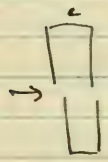
$l = 0$: $\int_{\theta_1}^{\theta_2} E_\theta \cdot r\,d\theta = \text{const} \ln \tan \theta/2 \Big|_{\theta_1}^{\theta_2}$
$\therefore$ maximum "potential difference" at center,
$\therefore$ electron thru center has maximum coupling.

Problem: Calculate "Q" of cavity $\int H_{tan}^2$.

Jan. 7 '42 Q for spherical cavity with holes:

$$Q \sim \frac{\int H^2 dV}{D \int H^2 d\sigma} \approx \frac{V}{AD} \sim \frac{R}{D}.$$

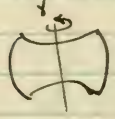




[for rectangular cavity, we had, $Q \sim \frac{c}{D}$ $R \gg c$, $\therefore$ Q much larger for spherical cavity.

[actually Q is considerably less than $\frac{R}{D}$; but still better than rectangular cavity]

Hansen & Rieckhofer also calculate for a cavity made out of sections of ellipse & hyperbola of rotation:



General theory of oscillations in a cavity — Condon, J. Appl. Ph. 1940-41.

Given a general cavity; $\underline{E}$ & $\underline{H}$ can then be calculated from Maxwell's equations & boundary conditions:

$$\begin{cases} c\nabla\times\underline{E} = -\dot{\underline{H}} \\ c\nabla\times\underline{H} = \dot{\underline{E}} \\ \nabla\cdot\underline{H} = 0 \\ \nabla\cdot\underline{E} = 0 \\ \underline{E}_{tan} = 0 \text{ at surface} \end{cases}$$

Introduce vector potential $\underline{A}$:

$$\underline{H} = \nabla\times\underline{A} \qquad \text{no scalar pot.}$$

$$\underline{E} = -\frac{1}{c}\dot{\underline{A}} \qquad \text{- no charges.}$$

Spatial dependence of A is same as E for any normal mode (i.e. (harmonic time dependence.))

$\therefore\ \nabla\cdot\underline{A} = 0$ for normal modes.

A satisfies: $\nabla^2\underline{A} = +\frac{1}{c^2}\ddot{\underline{A}}$

for a normal mode $A_m(t) = A_m e^{i\omega_m t}$

$$\nabla^2 A_m = -k_m^2 A_m \qquad \text{with } -k_m = \frac{\omega_m}{c}.$$

$$\text{or } (\nabla^2 + k^2)A_m = 0.$$

subject to $\underline{A}_{tan} = 0$ at surface

Treat as wave-mechanical problem; A_m are wave-functions, k_m are eigenvalues.

) Are A_m orthogonal?

~~$\int \underline{A}_\ell \cdot \underline{A}_m \, dV$~~

$$\int (\underline{A}_\ell\cdot\nabla\times\nabla\times\underline{A}_m - \underline{A}_m\cdot\nabla\times\nabla\times\underline{A}_\ell)\,dV = (k_m^2 - k_\ell^2)\int \underline{A}_\ell\cdot\underline{A}_m\,dV$$

Left hand side can be made a surface integral;

$$\nabla\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot\nabla\times\underline{A} - \underline{A}\cdot\nabla\times\underline{B}$$

$$\therefore\ \nabla\cdot(\underline{A}_\ell\times\nabla\times\underline{A}_m) = \nabla\times\underline{A}_m\cdot\nabla\times\underline{A}_\ell - \underline{A}_\ell\cdot\nabla\times\nabla\times\underline{A}_m$$

$$\nabla\cdot(\underline{A}_m\times\nabla\times\underline{A}_\ell) = \nabla\times\underline{A}_m\cdot\nabla\times\underline{A}_\ell - \underline{A}_m\cdot\nabla\times\nabla\times\underline{A}_\ell$$

$\therefore$ Left side above $= \int (\underline{A}_\ell\times\nabla\times\underline{A}_m - \underline{A}_m\times\nabla\times\underline{A}_\ell)_n\,dS$

To have normal comp of $(A \times B)$, need tangential components of $A \times B$; since $\underline{A}_{m\,tan} = 0 = \underline{A}_{\ell\,tan}$, the surface integral vanishes;

Hence $(k_m^2 - k_\ell^2) \int \underline{A}_\ell \cdot \underline{A}_m = 0$

Hence $\int \underline{A}_\ell \cdot \underline{A}_m = 0$ unless $k_m = k_\ell$.

i.e. two non-degenerate eigenfunctions are orthogonal.

[Can, in case of degeneracy, find solutions which are orthogonal — linear combinations of original solutions.]

[In the cases we have treated, we had orthogonal degenerate eigenf'ns: spherical cavity, P_ℓ^m are degenerate with resp. to m, but P_ℓ^m are orthogonal].

Normalization: we will use $\int_V A_m^2 \, dV = V$

Example: Rectangular cavity:

We had $E_x = C_x \cos\frac{\pi n_1 x}{L_1} \sin\frac{\pi n_2 y}{L_2} \sin\frac{\pi n_3 z}{L_3} = A_x$

L_1, L_2, L_3 = edges of cavity, n_1, n_2, n_3 integers.

then $\int A^2 dV = C^2 V \cdot \frac{1}{8}$ $\qquad C^2 = C_x^2 + C_y^2 + C_z^2$.

$\therefore C^2 = 8$ to get desired normalization

[also $\underline{C} \cdot \underline{k} = 0$].

If $n_1 = 0$ $\quad A_y = 0 = A_z$

and $\int A^2 dV = \frac{1}{4} C_x^2 V$ & $C_x = 2$.

$\therefore$ normalized eigenf'n is $A_x = 2 \sin\frac{\pi n_2 y}{L_2} \sin\frac{\pi n_3 z}{L_3}$

Energy: $W = \frac{1}{8\pi} \int (E^2 + H^2) \, dV$.

Vector potential can be expanded in terms of normal modes:

$$\underline{A} = \sum_m p_m(t) \underline{A}_m(\underline{r}).$$

$$-\frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\nabla^2 A$$

$$\therefore \quad \frac{1}{c^2}\sum \ddot{p}_m(t)\, A_m = \sum p_m \nabla^2 A_m$$

$$= \sum p_m(-k^2 A_m)$$

Since A_m are arbitrary, A_m must have same coeff. on each side: $\frac{1}{c^2}\ddot{p}_m = -k^2 p_m$

$$\ddot{p}_m + \omega^2 p_m = 0$$

whence $p_m = \overset{\omega}{\sin}\, \omega_m t$.

Dissipation term may be added — proportional to $\dot{p}_m$: then get:

$$\ddot{p}_m + \frac{\omega_m}{Q_m}\dot{p}_m + \omega_m^2 p_m = 0$$

To show this is correct form, will calculate energy + energy loss.

This is for free oscillation; will also discuss forced oscillation (loop, or electron beam, thru cavity).

Now we have:

$$\underline{E} = -\frac{1}{c}\sum \frac{dp_m}{dt}\cdot \underline{A}_m$$

$$\underline{H} = \sum p_m \operatorname{curl} \underline{A}_m$$

then

$$W = \frac{1}{8\pi}\sum_\ell \sum_m \left(\dot{p}_\ell \dot{p}_m \int dV\, A_\ell \cdot A_m + p_\ell p_m \int dV \operatorname{curl} A_m \cdot \operatorname{curl} A_\ell\right)$$

$$= \frac{1}{8\pi}\sum_\ell \sum_m \left(\dot{p}_\ell \dot{p}_m \frac{\delta_{\ell m}}{c^2} V + p_\ell p_m \int dV\, A_\ell \cdot \operatorname{curl}\operatorname{curl} A_m\right)$$

$$\int dV\, A_\ell \cdot k^2 A_m$$

$$\therefore W = \frac{1}{8\pi}\sum_m \left(\frac{\dot{p}_m^2 + \omega_m^2 p_m^2}{c^2}\right) V = \frac{V}{8\pi c^2}\sum \left(\dot{p}_m^2 + \omega^2 p_m^2\right) \underbrace{k^2 V \cdot \delta_{\ell m}}$$

Jan. 9, '42 We found solutions $\underline{A}_m$ such that $\int \underline{A}_\ell \cdot \underline{A}_m \, dV = V \delta_{\ell m}$

We expanded $\underline{A} = \sum p_m(t) \, \underline{A}_m(s)$

and found that p_m satisfy $\ddot{p}_m + \omega^2 p_m = 0$.

then $\underline{E} = -\frac{1}{c} \sum \dot{p}_m \underline{A}_m$, $H = \sum p_m \nabla \times \underline{A}_m$.

We found further: $\int (\nabla \times A_\ell) \cdot (\nabla \times A_m) \, dV = k_m^2 V \delta_{\ell m}$

Hence $\nabla \times \underline{A}$ is as good an eigenf'n as A.

If we write $\nabla \times \underline{A}_\ell = k_\ell \underline{F}_\ell$

then $\int \underline{F}_\ell \cdot \underline{F}_m \, dV = V \delta_{\ell m}$ just like A's.

Now $\nabla \times \underline{F}_\ell = \frac{1}{k_\ell} \nabla \times \nabla \times A_\ell = \frac{1}{k_\ell} \cdot k_\ell^2 \underline{A} = k_\ell A$.

So F & A have a symmetrical, reciprocal relationship

We had also $W = \frac{1}{8\pi} \int (E^2 + H^2) \, dV = \frac{V}{8\pi} \sum_m \left(\frac{1}{c} \dot{p}_m^2 + k_m^2 p_m^2 \right)$

$$= \frac{V}{8\pi c^2} \sum \left(\dot{p}_m^2 + \omega^2 p_m^2 \right)$$

If damping, $\ddot{p}_m + \frac{\omega_m}{Q_m} \dot{p}_m + \omega^2 p_m = 0$

if no damping: $\therefore p_m = \begin{matrix} \cos \\ \sin \end{matrix} \omega_m t$

$$\overline{\dot{p}_m^2} = \omega_m^2 \overline{p_m^2}$$

$$\therefore W_{el} = W_{mag}.$$

If E, H are given at $t=0$,

$$\left.\begin{aligned} \int \underline{E} \cdot \underline{A}_m \, dV &= -\frac{1}{c} \frac{dp_m}{dt} \cdot V \\ \text{and} \int \underline{H} \cdot \underline{F}_m \, dV &= k_m p_m V \end{aligned}\right\} \text{and can find initial value of } p\text{'s} + \dot{p}\text{'s}.$$

$$\text{or} \left\{ \begin{aligned} \dot{p}_m(0) &= \frac{c}{V} \int \underline{E}(0) \cdot \underline{A}_m \, dV \\ p_m(0) &= \frac{1}{k_m V} \int \underline{H}(0) \cdot \underline{F}_m \, dV \end{aligned} \right.$$

Dissipation. $\ddot{p} + \frac{\omega_m}{Q_m}\dot{p} + \omega_m^2 p = 0$

$$\ddot{p}\dot{p} + \frac{\omega_m}{Q_m}\dot{p}^2 + \omega_m^2 p\dot{p} = 0$$

$$\therefore \tfrac{1}{2}\dot{p}^2 + \int \frac{\omega_m}{Q_m}\dot{p}^2 dt + \tfrac{1}{2}\omega_m^2 p^2 = C.$$

$$\therefore \quad W = \tfrac{1}{2}(\dot{p}^2 + \omega_m^2 p^2) = C - \frac{\omega_m}{Q_m}\int \dot{p}^2 dt$$

so energy decreases with time.

$$\therefore \frac{dW}{dt} = -\frac{\omega_m}{Q_m}(\dot{p})^2 = P \quad \text{power phs}$$

~~unit time~~ consumption

We had $\quad P \sim D\nu \int H^2 d\sigma$

But $\quad \int H^2 d\sigma = \sum_{l,m} p_l p_m \int (\nabla\times A_l)(\nabla\times A_m)\, d\sigma$

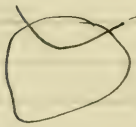
$$= \sum_m p_m^2 h_m^2 \int F_m^2 d\sigma$$

For time averages: $\quad \overline{p_m^2} h_m^2 = \frac{1}{c^2}\overline{\dot{p}^2}$

$$\therefore \cancel{\int H^2 d\sigma} = -\frac{\omega_m}{Q_m}\overline{\dot{p}^2} = D\nu \sum_m \overline{p_m^2} h_m^2 \int F_m^2$$

$$\therefore \frac{\omega_m}{Q_m} = D\nu \sum_m h_m^2 \int F_m^2 d\sigma$$

Excitation.



→ conductor thru cavity.

use current ~~of~~ with freq. close to cavity freq.

Condon does this:

$$\nabla\times \underline{H} = \frac{1}{c}\dot{\underline{E}} + 4\pi \underline{j} \qquad \underline{j} \text{ in e.m.u.}$$

$$\nabla\times\nabla\times \underline{A} = \cancel{\nabla\times E}$$

$$= -\frac{1}{c^2}\ddot{A} + 4\pi j$$

put in expansion: $\quad \sum_m (\nabla\times\nabla\times A_m + \frac{1}{c^2}\ddot{A}_m \ddot{p}_m) = 4\pi j$

$$\therefore \quad \sum_m p_m k_m^2 A_m + \frac{1}{c^2}\ddot{p}_m A_m = 4\pi \underline{j} .$$

Expand j $= \sum_m A_m \frac{\int dV\, j \cdot A_m}{V}$.

Hence $p_m k_m^2 + \frac{1}{c^2}\ddot{p}_m = \frac{4\pi}{V}\int \underline{j}\cdot\underline{A}_m\, dV$.

or $\ddot{p}_m + \omega_m^2 p_m = \frac{4\pi c^2}{V}\int \underline{j}\cdot\underline{A}_m\, dV$.

If also dissipation : $\ddot{p}_m + \frac{\omega_m}{Q_m}\dot{p} + \omega_m^2 p_m = \frac{4\pi c^2}{V}\int \underline{j}\cdot\underline{A}_m\, dV$

def: $= L p_m$

If current is in a wire of small cross section, or small beam of electrons Here can integrate first over cross section of beam, then along whole beam ;
(gives total current)

$$\therefore L p_m = \frac{4\pi c^2}{V} I \int_{\text{wire}} \underline{A}_m \cdot d\underline{s} \quad = (\quad) M_m$$

if loop is completed along outside, $\oint \underline{A}_m \cdot d\underline{s} = \int (\nabla \times \underline{A}_m)\cdot d\sigma$
(contribution from outside = 0) $= \int H_m \cdot d\sigma$
= flux thru loop due to unit excitation of mode m of cavity

If $I = I_0 e^{i\omega t}$
where $\omega \neq$ any ω_m

From theory of forced oscillation :

$$p_m = \frac{4\pi c^2}{V} M_m I_0 e^{i\omega t} \frac{1}{\omega_m^2 - \omega^2 + \frac{i\omega\omega_m}{Q_m}}$$

+ resonance at cavity frequencies.

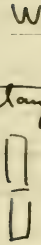
For general excitation $I(t)$.
get thus from theory of 2nd order eq'ns :

$$p = \frac{4\pi c^2}{V} M_m \int_{-\infty}^{t} I(t') \sin\omega_m (t-t') e^{-\frac{\omega_m}{iQ}(t-t')}\, dt' .$$

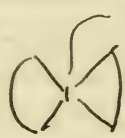
Jan 12, '42 For excitation : $\ddot{p}_m + \frac{\omega_m}{Q_m}\dot{p}_m + \omega_m^2 p_m = \frac{4\pi c^2 I}{V}\int \underline{A}_m \cdot d\underline{s}$

Want $\int \underline{A}_m \cdot d\underline{s}$ to be as large as possible

rectangular cavity:



integral relatively small.



integral large
field large
at center
$\left[A \sim \frac{\cos kr}{r} \sin\theta\right.$

I impedance produced by cavity in circuit of exciter.

each mode is excited:

$$p_m = \frac{4\pi c^2 I \int \underline{A}_m \cdot d\underline{s}}{\omega_m^2 - \omega^2} e^{i\omega t}$$

now $E = -\frac{1}{c}\sum \dot{p}_m \underline{A}_m = -\frac{1}{c}\sum i\omega p_m \underline{A}_m$

then $\frac{\int \underline{E}\cdot d\underline{s}}{I}$ = impedance = R

$$R = \frac{1}{cI}\sum \dot{p}_m i\omega \int \underline{A}_m \cdot d\underline{s}$$

$$= -\frac{4\pi c^2 I}{cI} i\omega \sum \frac{\left(\int \underline{A}_m \cdot d\underline{s}\right)^2}{\omega_m^2 - \omega^2}$$

or $R_{\text{[illegible] (should be)}} = +i\omega 4\pi \sum \frac{\left(\int \underline{A}_m \cdot d\underline{s}\right)^2}{\omega_m^2 - \omega^2}$.

for $\omega \ll \omega_m$, $R \sim +i\omega$ inductance

$\omega \sim \omega_m$ resonances.

Bunching :

~~two~~ deviation :

(+) energy transfer to electron in 1st cavity :

$\xrightarrow{\quad}$ Ω

$a V \sin \omega t$.

$$E = p_m e^{i\omega t} \underline{A}_m(r)$$

$$V = \int \underline{E} \cdot d\underline{s} = p_m e^{i\omega t_0} \int A_{m_x}(x)\, e^{i\omega x/v}\, dx$$

retardation -

Simplest case : $A_{m_x} = \text{const.}$

hence $A_{m_x} \int_{-b/2}^{b/2} e^{i\omega x/v}\, dx$

$$= A_{m_x} \cdot \frac{e^{i\omega \frac{b}{2v}} - e^{-i\omega \frac{b}{2v}}}{\frac{i\omega}{v}}$$

$$= A_{m_x} \cdot b \cdot \frac{\sin \frac{\omega b}{2v}}{\frac{\omega b}{2v}} .$$

$\therefore$ want $\frac{\omega b}{2v} < \pi/2$ ~~if [illegible]~~

$\frac{b}{v} < \pi/\omega = \frac{1}{2}$ period

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_0^2 + e V \sin \omega t$$

$$v^2 = v_0^2 + \frac{2eV}{m} \sin \omega t$$

or $v = \left(v_0^2 + \frac{2eV}{m} \sin \omega t\right)^{1/2} \doteq v_0 + \frac{eV}{m v_0} \sin \omega t$

(if ω small)

$$x(t') = v(t' - t)$$

Second cavity at $x = a$. $\therefore t' - t = \frac{a}{v} = \frac{a}{v_0} - \frac{a e V}{m v_0^3} \sin$

Excitation of second cavity :

$$p_2(t'') = \int^{t''} I(t') \sin\omega(t'' - t')\, e^{-\frac{\omega}{Q}(t'' - t')}\, dt'$$

if $I(t')$ is constant, $p_2(t'') \sim 0$.

but $I(t')$ is not constant, because of bunching –
first many, then no electrons reach it – in rapid succession

$I(t')\,dt' = I(t)\,dt$ but $I(t) \sim$ const. (1st cavity)

$$\therefore p_2(t'') = \int_0^{t''} I(t)\,dt \,\sin\omega\left(t''-t-\frac{a}{v_0}+\frac{aeV}{mv_0^3}\sin\omega t\right)\cdot e^{-\frac{\omega}{Q}\left(t''-t-\frac{a}{v_0}\right)}$$

write integral as:

$$\sin\omega\left(t''-\frac{a}{v_0}\right)\cos\omega\left(t-\frac{aeV}{mv_0^3}\sin\omega t\right)$$
$$-\cos \quad '' \quad \sin \quad ''$$

exponential is essentially constant –

$\therefore$ need

$$\int_{\omega t=0}^{2\pi}\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\omega\left(t-\frac{aeV}{mv_0^3}\sin\omega t\right)d(\omega t)$$

$$=\left\{\begin{matrix}2\pi\\ 0\end{matrix}\right. J_1\left(\frac{aeV\omega}{mv_0^3}\right)$$

$$\therefore p_2(t'') = I\sin\omega\left(t''-\frac{a}{v_0}\right)J_1\left(\frac{aeV\omega}{mv_0^3}\right)\cdot\frac{Q}{\omega^2}$$

$\therefore$ excitation of second cavity $\sim I$ (elect. current)
$\sim \sin\omega t''$ same freq.
$\sim Q$ so large Q desirable
$\sim J_1$

J_1 has max when arg. $= 1.83$

$$\therefore aeV\omega = 1.83\, mv_0^3$$

gives max. or: $a = \dfrac{1.83\, mv_0^2}{eV}\cdot\left(\dfrac{v_0}{\omega}\right)$

$$\simeq \frac{E_0}{eV}\cdot b\cdot 1.1$$

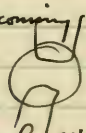
But then p_2 depends on I,
not on field in 1st cavity –
so arbitrary amplification.

$$\therefore \frac{a}{b} \sim \frac{E_0}{eV}.$$

(but need signal large compared with shot effect noise)

Jan 14, '42 As detector: Let signal & a local oscillator both excite cavity:

Incoming ω_2



local: ω_1

$\omega_1 \approx \omega_2$. Now we have, in previous equation

$$e V_1 \sin \omega_1 t + e V_2 \sin \omega_2 t$$

$$= e V_1 \sin \omega_1 t + e V_2 \left[\sin \omega_1 t \cos(\omega_2 - \omega_1 t) + \cos \omega_1 t \sin \right.$$

$$= e \sin \omega_1 t \left[V_1 + V_2 \cos(\omega_2 - \omega_1) t \right] + \cos \omega_1 t \sin(\omega_2 - \omega_1$$

$$= e \sin\left[\omega_1 t + \frac{V_2}{V_1} \sin(\omega_2 - \omega_1) t\right] \left[V_1 + V_2 \cos(\omega_2 - \omega_1) t\right]$$

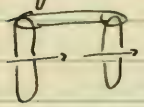
And the integral becomes: $J_1\left(\frac{a e \omega}{m v_0^3}\left(V_1 + V_2 \cos(\omega_2 - \omega_1) t\right)\right) \sin \omega$

Make $J_1'\left(\frac{a e \omega}{m v_0^3} V_1\right)$ a maximum.

then get $J_1\left(\frac{a e \omega V_1}{m v_0^3}\right) + \frac{a e \omega}{m v_0^3} V_2 \cos(\omega_2 - \omega_1) t\, J_1'$

constant ↗ detectable ↖

Klystron:



what effect starts it; afterwards, second cavity is coupled to first, which bunches electrons which excite second cavity.

Emission of Radiation.

Cf. Stratton esp. applications

Use φ, $\underline{A}$, scalar + vector potential resp.

then $\underline{H} = \nabla \times \underline{A}$, $\underline{E} = -\frac{1}{c}\dot{\underline{A}} - \nabla\varphi$

[then $\nabla \times \underline{E} = -\frac{1}{c}\nabla\times\dot{\underline{A}} - \nabla\times\nabla\varphi = -\frac{1}{c}\dot{\underline{H}}$ ✓

and $\nabla\cdot H = \nabla\cdot\nabla\times A \equiv 0$ automatically ✓]

We also have: $\nabla\cdot\underline{E} = 4\pi\rho$, $\nabla\times H = +\frac{1}{c}\dot{\underline{E}} + \frac{4\pi}{c}\underline{j}$

these give: $-\frac{1}{c}\nabla\cdot\dot{\underline{A}} - \nabla\cdot\nabla\varphi = 4\pi\rho$; $\nabla\times\nabla\times A = -\frac{1}{c^2}\ddot{\underline{A}} - \frac{1}{c}\nabla\dot{\varphi} + \frac{4\pi}{c}\underline{j}$

If we impose the condition $\left\{\begin{array}{l}\nabla\cdot\underline{A} + \frac{1}{c}\dot{\varphi} = 0,\\ \nabla\cdot\underline{j} + \frac{\partial\rho}{\partial t} = 0\end{array}\right.$

then $\left\{\begin{array}{l}\nabla^2\varphi - \frac{1}{c^2}\ddot{\varphi} = -4\pi\rho\\ \nabla^2\underline{A} - \frac{1}{c^2}\ddot{\underline{A}} = -\frac{4\pi}{c}\underline{j}\end{array}\right.$

Which have sol'ns: $\varphi(\underline{r}, t) = \int d\underline{r}' \, \frac{\rho(\underline{r}', t - \frac{|\underline{r}-\underline{r}'|}{c})}{|\underline{r}-\underline{r}'|}$

"retarded" potential.

$$\underline{A}(\underline{r}, t) = \int d\underline{r}' \, \frac{\underline{j}(\underline{r}', t - \frac{|\underline{r}-\underline{r}'|}{c})}{c|\underline{r}-\underline{r}'|}$$

Can show from these that $\nabla\cdot\underline{A} + \frac{1}{c}\dot{\varphi} = 0$, so that condition was OK.

Since $\underline{j}$ & ρ are connected, should be able to get φ from $\underline{A}$; this was done in older literature by introducing $\underline{\Pi}$ - the Hertzian vector; then $\underline{A} = \dot{\underline{\Pi}}$, $\varphi = \nabla\cdot\underline{\Pi}$

We will assume ~~immediately~~ a harmonic time dependence: $e^{-i\omega t}$ [then waves travelling, say, in $+z$ direction have e^{+ikz}].

Then $\left\{\begin{array}{l}\underline{j}(\underline{r}', t) = \underline{j}_0(\underline{r}')\, e^{-i\omega t}\\ \rho(\underline{r}', t) = \rho_0(\underline{r}')\, e^{-i\omega t}\end{array}\right.$ then $\nabla\cdot\underline{j}_0 - i\omega\rho_0 = 0$

$$\rho_0 = \frac{1}{i\omega}\nabla\cdot\underline{j}_0$$

this gives : $\varphi(\underline{r}, t) = \int \frac{dv'}{|r-r'|} \rho_0(\underline{r}') e^{-i\omega\left[t - \frac{|r-r'|}{c}\right]}$

$$= \frac{e^{-i\omega t}}{i\omega} \int \frac{dv'}{|r-r'|} \underline{\nabla} \cdot \underline{j}_0 \, e^{ik|r-r'|} \qquad k = \frac{\omega}{c}.$$

and $\underline{A}(\underline{r}, t) = \frac{1}{c} e^{-i\omega t} \int \frac{dv'}{|r-r'|} \underline{j}_0(\underline{r}') e^{ik|r-r'|}$.

$$\text{i.e.} = \frac{e^{-i\omega t}}{i\omega} \int_V dv' \, \underline{j}_0(\underline{r}') \cdot \text{grad}_r \frac{e^{ik|r-r'|}}{|r-r'|}$$

$$\cong \nabla \cdot \left\{ \int dv' \, \underline{j}_0(\underline{r}') \cdot \frac{e^{ik|r-r'|}}{|r-r'|} \right.$$

i.e. $\varphi = \frac{1}{ik} \underline{\nabla} \cdot \underline{A}$.

[This comes also from $\nabla \cdot A + \frac{1}{c} \dot{\varphi} = 0$].

If currents are confined to a region small comp. with wavelength, get practically useful result for points near it. i.e. essentially static fields.

At large distances, results again simple - so called "wave zone"

At intermediate regions - rather complicated.

Wave zones : $r \gg r'$ Field at distance $\gg$ dimension of radiator.

also $r \gg \lambda$.

then neglect $\underline{r}'$ comp. with $\underline{r}$ in denominator;

in exp. write : $|\underline{r} - \underline{r}'| = r - \underline{s} \cdot \underline{r}'$ $\underline{s}$ = unit vector in direction of r

then $\underline{A}(\underline{r}, t) = \frac{e^{-i\omega t}}{cr} \int dv' \, \underline{j}_0(\underline{r}') e^{ik(r - \underline{s} \cdot \underline{r}')}$

$$= \frac{1}{cr} e^{i(kr - \omega t)} \int dv' \, \underline{j}_0(\underline{r}') e^{-ik \underline{s} \cdot \underline{r}'}.$$

Define: $\underline{J}(\underline{s}) = \int d\underline{r}'\, \underline{j}_0(\underline{r}')\, e^{-ik\underline{s}\cdot\underline{r}'}$

Then $\underline{A}(\underline{r},t) = \frac{1}{cr}\, e^{i(kr-\omega t)}\, \underline{J}(\underline{s})$.

Jan 16, '42 then $\varphi = \frac{1}{ik}\nabla\cdot\underline{A} = \frac{1}{ik}\cdot\frac{e^{-i\omega t}}{cr}(\nabla e^{ikr})\cdot\underline{J}$ [assuming e^{ikr} strongest r-depen

$$= \frac{1}{ik}\,\frac{e^{-i\omega t}}{cr}\cdot ik\,\underline{s}\cdot\underline{J}\, e^{ikr}$$

$$= \frac{e^{ikr-i\omega t}}{cr}\,\underline{s}\cdot\underline{J} = \underline{s}\cdot\underline{A} .$$

And $\underline{H} = \underline{\nabla}\times\underline{A} = \frac{1}{cr}\, e^{-i\omega t}(\nabla e^{ikr})\times\underline{J}$

$$= \frac{1}{cr}\, e^{i(kr-\omega t)}\cdot ik\, s\times\underline{J}$$

$$= \frac{ik}{cr}\, e^{i(kr-\omega t)}\, \underline{s}\times\underline{J} .$$

Likewise $\underline{E} = -\frac{1}{c}\dot{\underline{A}} - \nabla\varphi = -\frac{1}{c}\cdot -i\omega\underline{A} - ik\,\underline{s}\,\underline{s}\cdot\underline{A}$

$$= ik\,\underline{s}\times(\underline{A}\times\underline{s}) .$$

Hence E, H both $\perp$ $\underline{s}$, direction of propagation.
also $E \perp H$, and in same phase.
outgoing spherical waves; phase propagated outward

If $\underline{J}$ is measured in amperes:

$$\underline{E} = \frac{ik}{10}\left\{\frac{e^{i(kr-\omega t)}}{r}\,\underline{s}\times(\underline{J}\times\underline{s})\right\} \text{ e.s.u.}$$

$$= 30ik\,\{\quad\} \text{ volts/cm.}$$

Energy flux: Poynting vector $\underline{S} = \frac{c}{4\pi}\,\underline{E}\times\underline{H}$

$$\overline{\underline{S}} = \frac{c}{4\pi}\cdot\frac{1}{2}\left|\underline{E}\times\underline{H}^*\right|$$

$$\underline{E} \times \underline{H}^* = k^2 \cdot \frac{1}{r^2} \left[\underline{s} \times (\underline{J} \times \underline{s}) \right] \times \left(\underline{J}_a \times \underline{s} \right)$$

$$= \frac{k^2}{r^2} \, \underline{s} \, |\underline{J}_a \times \underline{s}|^2 .$$

$$\therefore \quad \underline{S} = \frac{c}{8\pi} \frac{k^2}{r^2} \overline{(\underline{J}_a \times \underline{s})^2} \cdot \underline{s} \qquad \text{erg/cm}^2\text{/sec.}$$

($J \sim \omega \cdot$ Dipole moment)

where $S \sim \omega^4$ (Dip. mom.)^2

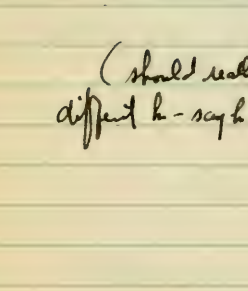
Per unit solid angle $d\alpha$:

$$\frac{c}{4\pi} d\alpha \; k^2 \overline{(\underline{J}_a \times \underline{s})^2} \quad \text{erg/sec.}$$

$$\text{or} : \quad \frac{30}{4\pi} d\alpha \; k^2 \overline{(\underline{J}_a \times \underline{s})^2} \quad \begin{cases} \text{joule/sec} \\ \text{watt.} \end{cases}$$

<u>Examples</u>

Quarter wave antenna



$$j_0 = j_z = \overbrace{\text{const.}}^{a} \cos kz \qquad k = \frac{2\pi}{\lambda}$$

$$kl = \pi/2 \; ; \; l = \lambda/4$$

then $\quad J = a \int_0^l dz \cos kz \, e^{-ik s \cdot r'}$

(should really have different k - say k' for antenna)

$$= a \int_0^l \cos kz \, e^{-ikz\cos\theta} \, dz$$

$$= a/2 \int_0^l \left[e^{ikz(1-\cos\theta)} + e^{-ikz(1+\cos\theta)} \right] dz$$

$$= \frac{a}{2ik} \left. \frac{e^{ikz(1-\cos\theta)}}{1-\cos\theta} \right|_0^l - \left. \frac{e^{-ikz(1+\cos\theta)}}{1+\cos\theta} \right|_0^l$$

$$= \frac{a}{2ik} \left[\frac{e^{ikl} \cdot e^{-ikl\cos\theta} - 1}{1-\cos\theta} - \frac{e^{-ikl} e^{ikl\cos\theta} - 1}{1+\cos\theta} \right]$$

$$= \frac{a}{2ik} \left[\frac{(1+\cos\theta)(e^{ikl} e^{-ikl\cos\theta} - 1) - (1-\cos\theta)(e^{-ikl} e^{ikl\cos\theta} - 1)}{1-\cos^2\theta} \right]$$

$$= \frac{a}{2ih}\left[\frac{e^{-ikl\cos\theta}\left[e^{ikl}(1+\cos\theta) - e^{-ikl}(1-\cos\theta)\right]+2}{\sin^2\theta}\right]$$

$$= \frac{a}{2ih}\left[\frac{e^{-ikl\cos\theta}\left[2i\sin kl + 2\cos\theta\cos kl\right]+2}{\sin^2\theta}\right]$$

but $kl = \pi/2$

$$\therefore J = \frac{a}{2ik}\left[\frac{e^{-\frac{i\pi}{2}\cos\theta}\left[2i\right]+2}{\sin^2\theta}\right]$$

$$= \frac{a}{ik\sin^2\theta}\left(i e^{-\pi/2\, i\cos\theta} - \cos\theta\right)$$

$$\underline{J} \times \underline{s} = J\sin\theta = \frac{a}{ik\sin\theta}\left(i e^{-\pi/2\, i\cos\theta} - \cos\theta\right)$$

$$(\quad) = i\cos(\pi/2\cos\theta) + \sin(\pi/2\cos\theta) - \cos\theta$$

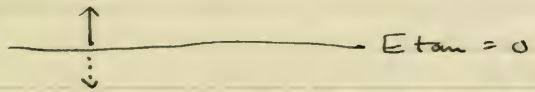
$$|\ |^2 = \cos^2(\pi/2\cos\theta) + \sin^2(\pi/2\cos\theta) - 2\cos\theta\sin(\quad) + \cos^2\theta$$

$$= 1 + \cos^2\theta - 2\cos\theta\sin(\pi/2\cos\theta)$$

$\cos\theta$ small: $\quad |\ |^2 = 1 - (\pi - 1)\cos^2\theta$

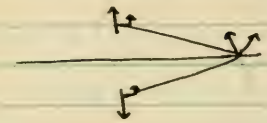
$$|\underline{J} \times \underline{s}|^2 = \frac{|\ |^2}{\sin^2\theta} = \frac{|\ |^2}{1-\cos^2\theta} = 1 - (\pi - 2)\cos^2\theta .$$

Earth is a conductor: ~~tangential~~ cmp. of E vanishes;
$\therefore$ take image antenna:



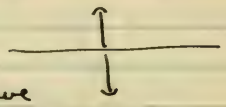
~~turns out~~ in above case that E at surface is normal to earth surface — but conductivity ~~attenuates~~ the wave.

Jan 19, 42 Antenna with earth: use mirror image:



charges symmetrical,
current not "

if on earth:



get half-wave antenna

then radiation becomes:

$$\frac{e^{ikl(1-\cos\theta)} - e^{-ikl(1-\cos\theta)}}{1-\cos\theta} - \frac{e^{-ikl(1+\cos\theta)} - e^{ikl(1+\cos\theta)}}{1+\cos\theta}$$

$kl = \pi/2$

$$= \frac{2i\sin[kl(1-\cos\theta)] + 2\cos\theta[\cos(kl(1-\cos\theta)] + 2i\sin(+}{\sin^2\theta} \quad \frac{}{+2\cos\theta[\cos -}$$

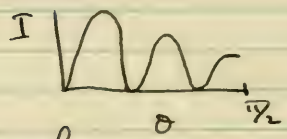
$$= \frac{2[\cos\theta\{\cos kl(-) + \cos kl(+)\} + i\sin\theta\{}{i^2\theta}$$

$$\sim \frac{2i\sin kl(1-\cos\theta)}{i^2\theta} \sim \frac{\cos(\pi/2\cos\theta)}{\sin^2\theta}$$

If $kl = (n+1/2)\pi$, get: $\dfrac{\cos((n+1/2)\pi\cos\theta)}{\sin^2\theta}$.

$I(\theta) = 0$ for $(n+1/2)\cos\theta = m + 1/2$

$\cos\theta = \dfrac{m+1/2}{n+1/2}$



this not good: want large $I(\theta)$ for $\theta \sim 90°$.

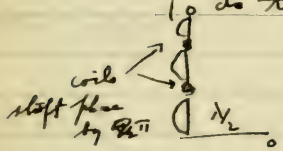
$\underline{J}(\underline{s}) = \int \underline{j}_0(\underline{r}')\, e^{ik\underline{s}\cdot\underline{r}'}\, d\underline{r}'$ if $\underline{s}$ is horizontal, $\underline{r}'$ is vertical

then $\underline{s}\cdot\underline{r}' = 0$, $\underline{J} = \int \underline{j}_0\, d$

i.e. make current have same phase everywhere along antenna — so 1/4 wave antenna is better than 3/4, etc —

But want long antenna to make J small for almost vertical $\underline{s}$; j_0 in phase with itself along antenna to make J large for horizontal $\underline{s}$.

To do this:



$i = \sin kz \qquad kz < 0 < \pi$

small contribution to $\int j_0 ds'$ from coil — neglect.

$i = \sin(kz - \lambda/2) \qquad \pi < kz < 2\pi$

Radiation is: $J = \sum_{n=-a}^{+a} \int_{-l/2}^{l/2} \cos kz' \, e^{-ik(nl+z')\cos\theta} dz'$ (z' measured from center)

$kl = \pi$ (2a+1 sections)

$$J = \sum_{n=-a}^{a} e^{-ikal\cos\theta} \int_{-l/2}^{l/2} \cos kz' \, e^{-ikz'\cos\theta} dz'$$

$$= \sum_{n=-a}^{a} e^{-ikal\cos\theta} \cdot \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

$$= \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \cdot e^{-i\pi\cos\theta a} \frac{e^{i(2a+1)\pi\cos\theta} - 1}{e^{\pi i\cos\theta} - 1}$$

$$= \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \cdot \frac{\sin\{\pi \cos(a+\frac{1}{2})\theta\}}{\sin(\frac{\pi}{2}\cos\theta)}$$

$$= \frac{\cot(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \cdot \sin\{\pi\cos[(a+\tfrac{1}{2})\theta]\}$$

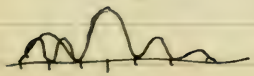
slowly varying — rapidly varying

Max for $\theta = \pi/2$

1st Min. for $\cos\theta = \frac{1}{a+\frac{1}{2}}$

2nd Max for $\cos\theta = \frac{3/2}{a+\frac{1}{2}}$

but much smaller.



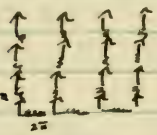
For larger no. of sections, more concentrated radiation (in $\theta = \frac{\pi}{2}$ direction)

This is still uniformly distributed in azimuth; if a directional antenna is desired, use two:

from in front, both are in phase; from the side, they are out of phase, + hence interfere destructively

this gives a fairly weak maximum.

add a third: ↑ ↑ ↑ get more pronounced maxima, + more minima; Make all these macroscopic antennas, + get only horizontal radiation in desired direction; in limit have a pinhole "diffraction pattern" of antennas:

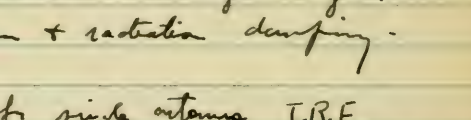


But now can no longer neglect residual + radiation damping.

Schelkunoff: radiation losses for single antenna I.R.E.
(points out that radiated energy cannot be treated like joule heat, which was done previously)

Jan 21 '42 (Perkins)

Array of antennas:



$$J = J_1 \sum_{m=1}^{M} \sum_{n=1}^{N} e^{-ik(s_y m l_y + s_z n l_z)}$$

where J_1 is radiation from single antenna.

$$J = J_1 \frac{\sin \frac{1}{2} M k l_y s_y}{\sin \frac{1}{2} k l_y s_y} \cdot \frac{\sin \frac{1}{2} N k l_z s_z}{\sin \frac{1}{2} k l_z s_z}$$

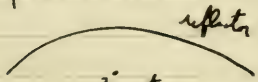
$l_y = l_z = \pi/k$ $\quad \therefore J = J_1 \dfrac{\sin \frac{M s_y}{2} \cdot \pi}{\sin \frac{\pi}{2} s_y} \cdot \dfrac{\sin \pi \frac{N s_z}{2}}{\sin \frac{\pi s_z}{2}}$.

Principal maxima at $\frac{1}{2} k l_y s_y = 0, \pi$

Want minima for $s_y = 1$ for directional effect (x-direction)

$\therefore \frac{1}{2} k l_y = \pi/2$. if $k l_y = 2\pi$ radiation in both x + y direction

reflectors also used for directional effects.



. . . . . reflecting waves.

Cantenna

long waves.

For long wave length, can shorten antenna by putting capacity on top

$$\underline{J} = \int \underline{j} \cdot e^{i k \underline{s} \cdot \underline{r}'} \, d\tau'$$

expand exp: $\quad e^{i k \underline{s} \cdot \underline{r}'} = 1 + i k (\underline{s} \cdot \underline{r}') - \frac{k^2}{2} (\underline{s} \cdot \underline{r}')^2 \cdots$

For long waves, small apparatus, can neglect retardation

$e^{i k \underline{s} \cdot \underline{r}} \sim 1$; then $\underline{J} = \int \underline{j}_0 \, d\tau$ i.e. J is indep. of $\underline{s}$.

~~then $\underline{E} \sim \underline{J} \times \underline{s} \sin\theta$ and $S^2 \sim s^{-2} \theta$~~

Poynting vector $\underline{S} \sim |\underline{J} \times \underline{s}|^2 \cdot \underline{s} \sim \sin^2\theta$

For long λ, if $\int \underline{j}_0 \, d\tau = 0$, radiation is much reduced; then must calculate the second term — "Quadrupole" radiation —

$J_q \sim J_d \, \kappa r$ $\quad$ r - dimension of radiating system.

$S_q \sim S_d \, (\kappa r)^2$.

$\underline{J}_d = \int \underline{j}_0 \, \underline{s}\cdot\underline{r} = \underline{\underline{A}}\,\underline{s}$ $\quad$ (A a dyadic).

$A_{ik} = \int j_{0i} \, r_k \; \left(\neq \int j_{0k} \, r_i \right)$ not symmetric but can be separated into sym. & unsym. parts -

$A = B + C$. $\quad$ $B_{ik} = \frac{1}{2} \int (j_{0i} r_k + j_{0k} r_i)$

$C_{ik} = \frac{1}{2}$ " - "

$C = \frac{1}{2} \int j \times r$ magnetic moment of magnetic dipole.

Two dipoles in opposite direction but displaced slightly give quadrupole radiation.

Similar to oscillations in a sphere:

E waves, $E_r \neq 0$ $H_r = 0$; H waves, $H_r \neq 0$, $E_r = 0$.

$\rightsquigarrow$ $E_r = P_{lm}(\cos\theta) \, e^{im\varphi} \, f(r)$.

$f'' + \frac{2}{r} f' + \left(\kappa^2 - \frac{l(l+1)}{r^2}\right) f = 0$

No solution for $l = 0$;

obtain $f \sim J_{l+\frac{1}{2}}(\kappa r) \, r^{-1/2}$

for outgoing waves, use Hankel f'n:

$f \sim H_{l+\frac{1}{2}}(\kappa r) \, r^{-1/2}$

$l = 1 \sim$ dipole rad.

$l = 2 \sim$ quadrupole rad.

$e^{i\kappa \underline{s}\cdot\underline{r}} \sim \frac{1}{r^{1/2}} \sum (2l+1) \, J_{l+\frac{1}{2}}(\kappa r) \, P_l(\underline{s}\cdot\underline{r})$

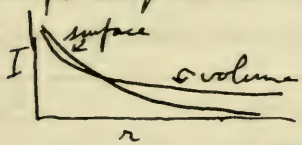
$= \sum_{l,m} J_{l+\frac{1}{2}}(\kappa r) \, P_{lm}(s) \, P_{lm}(r)$?

$\therefore \; J \sim \sum_{l,m} P_{lm}(s) \int j_0 \, d\tau \, \frac{1}{r^{1/2}} J_{l+\frac{1}{2}}(\kappa r) \, P_{lm}(r) \sim (\kappa r)^l$

Jan 23, '42
Last day!

Propagation.

antenna + mirror image (with current reversed) gives E normal to earth's surface, H along earth ⊥ to propagation. If earth has finite conductivity (not perfect conductor) get absorption; in wave guide case we found absorption per km $\sim \frac{\omega^{3/2}}{\sqrt{\sigma}}$. One finds a "surface wave" which is guided along earth much like waves in wave guide. Turns out that $I \sim \frac{1}{r}$, $\therefore E, H \sim \frac{1}{\sqrt{r}}$, so surface waves decrease in intensity slower than the volume wave. If absorption, however, $I \sim \frac{1}{r} e^{-\beta r}$ for surface wave; volume wave, $I \sim \frac{1}{r}$.



phases are different; may get interference between the two. Often a zone some distance from station, where practically no signal is heard.

Absorption increases with decreasing σ: on earth surface: sea water, little abs.; Fresh water, Wet earth, more abs.; Dry earth, most abs.

Spherical earth: Zenneck, Sommerfeld, Van der Pol. not of much significance.

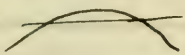
Reflection by ionized Heaviside layer ~

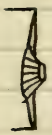
measure time between receiving direct & reflected signal at pt. near transmitter: is ~ 1000 μsec, indicating height ~ 150 km of layer.

Heaviside layer is due to ultra-violet in sun's radiation which ionizes N_2 & O_2, giving + ions & electrons. Recombination of ions takes a long time,

because (1) low pressure (2) small probability of capture of electron by ion; lifetime 10^{-8} sec for excited atom; 10^{-15} sec. is time it takes electron to traverse atom; so probability of radiation (i.e. capture of electron) is $\sim 10^{-7}$ per traversal. Usually have: triple collision; energy instead of going into radiation, goes into kinetic energy of another molecule; but this also rare because of low pressure.

At night the lowest layer disappears because of recombination. Lifetime $\sim$ a few hours in lower layers. (in upper layer, insignificant recombination in 12 hrs even)

Refecting layer can also absorb, especially if it has not a well defined edge  instead of .



In day, lower edge is more apt to be fuzzy in day time, hence better reception at night.

dielectric constant $\varepsilon = 1 - \frac{4\pi n e^2}{m\omega^2}$.

Scattering: in mountains reception bad because of scattering by mountains

Airplane detection; distinguish between $\lambda \gg a$ and $\lambda \ll a$ where $a \sim$ size of scattering object. In latter case, geometrical optics applies; scattering cross section $\sim a^2$. In former case, would have again scattering $\sim a^2$ if "monopole" waves ($l=0$); but for $l=1$, which actually is the case, scattering $\sim \frac{a^6}{\lambda^4}$. [polarization/moment in object $\sim E$ $\therefore M \sim E a^3$; and $E_{sc} \sim M k^2 \sim \frac{a^3}{\lambda^2}$ $\therefore$ intensity $\sim \frac{a^6}{\lambda^4}$]

$\lambda \sim a$ is desirable.

$$\underline{s}' = \underline{\underline{A}}\,\underline{s} =$$

$$= (\underline{p}\,\underline{q})\,s$$

$$= \underline{p}(\underline{q}\cdot\underline{s}) = (q_x s_x + q_y s_y + q_z s_z)\,p\,.$$

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